UNIVERSITÉ DE FRIBOURG / UNIVERSITÄT FREIBURG

Course given at the University of Fribourg (19 April 2010)

Introduction to Inversion in Geophysics

Dr. Laurent Marescot

Learning Objective and Agenda

Learning objective: get the basic understanding of inversion processes to be able to use geophysical software in a meaningful way

Agenda:

- Some basic definitions
- Inversion concepts
- Linear inversion: temperature example
- Non-linear inversion: seismic and geoelectric examples

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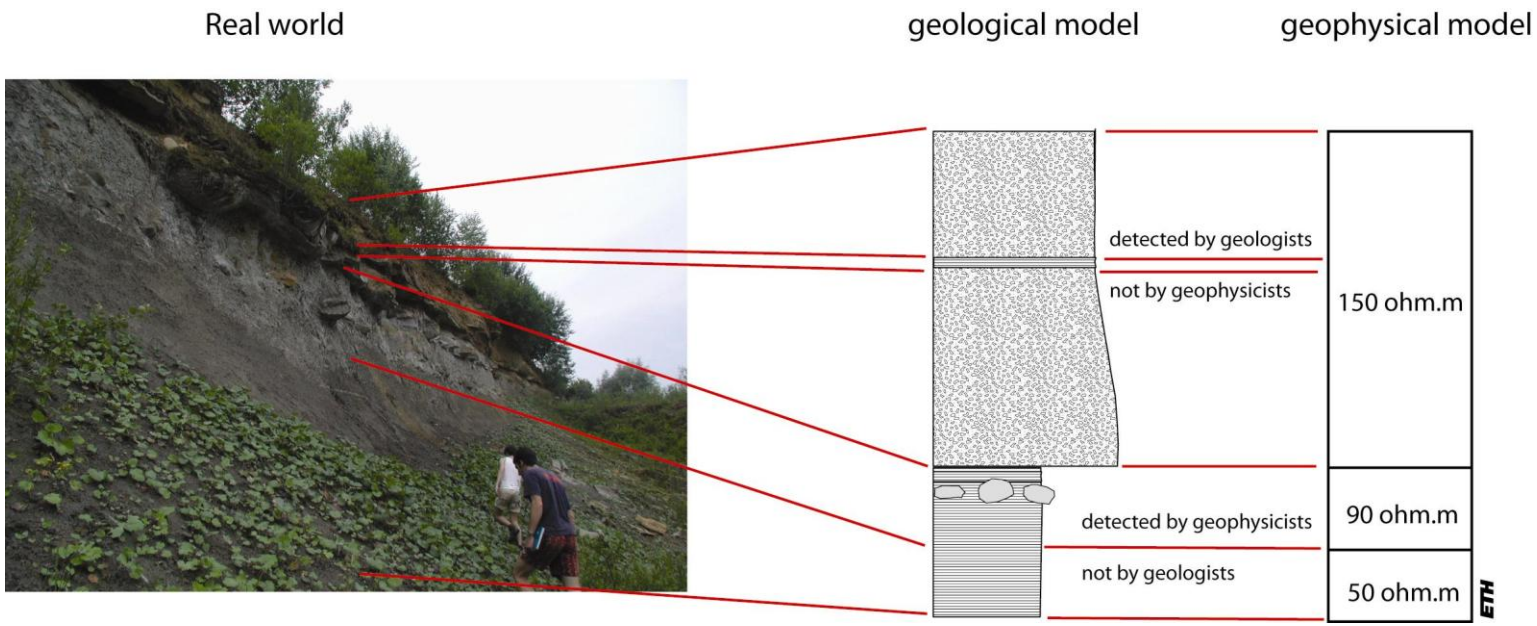
- Linear inversion: temperature example
- Non-linear inversion: seismic and geoelectric examples

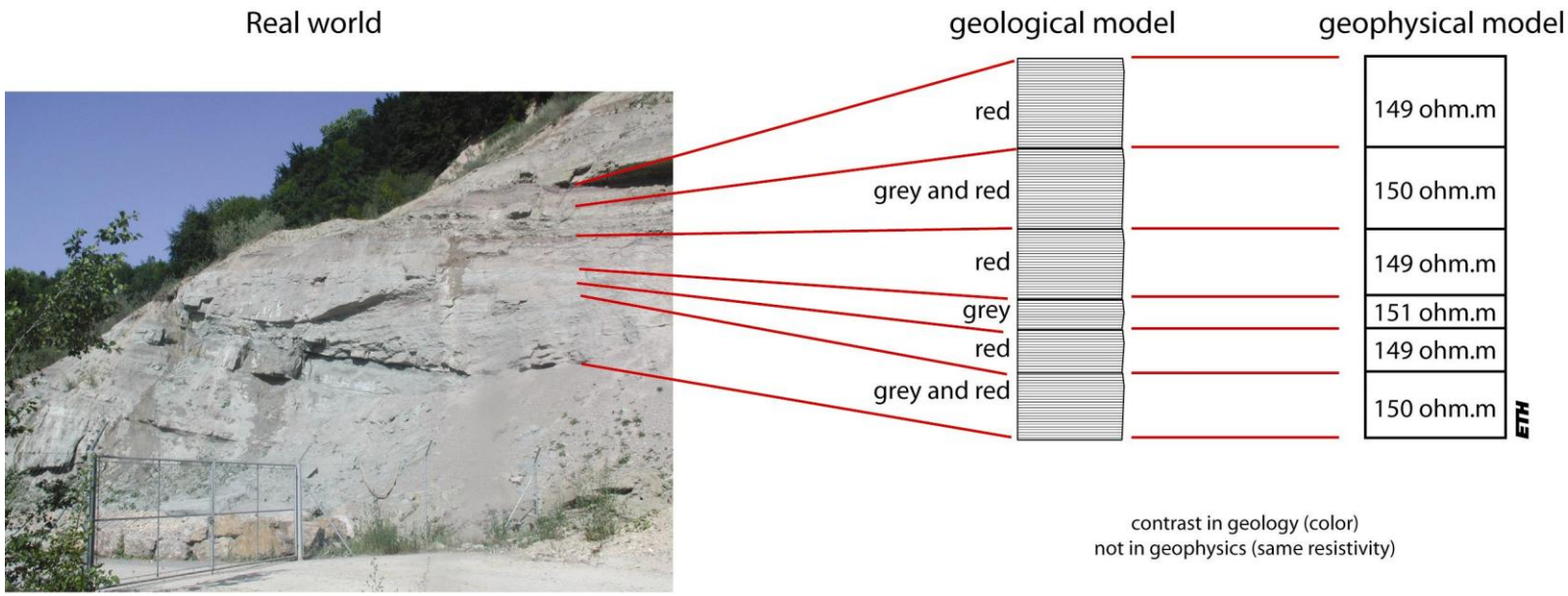
Definition: Model

A model is a simple and ideal view of a physical reality

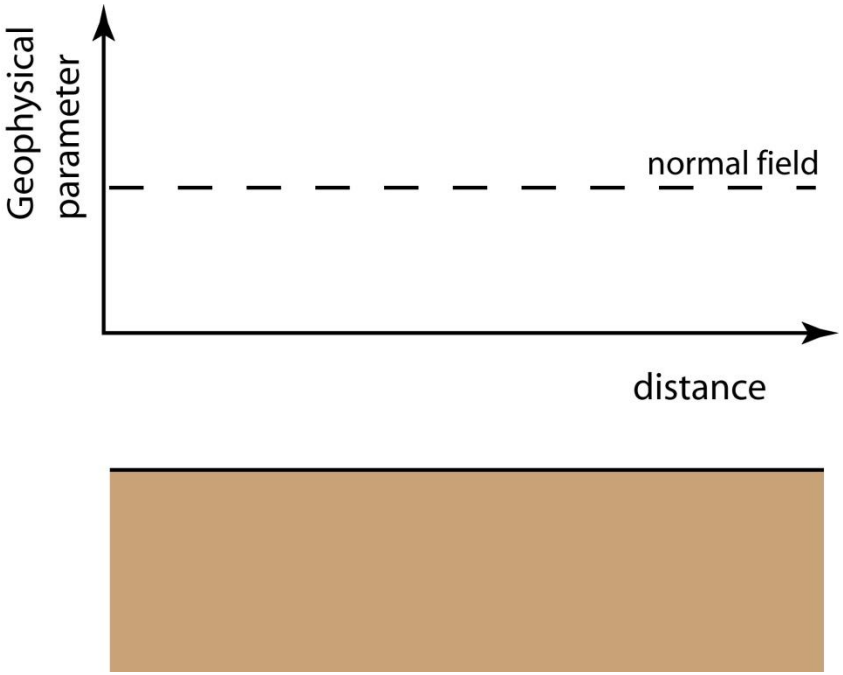


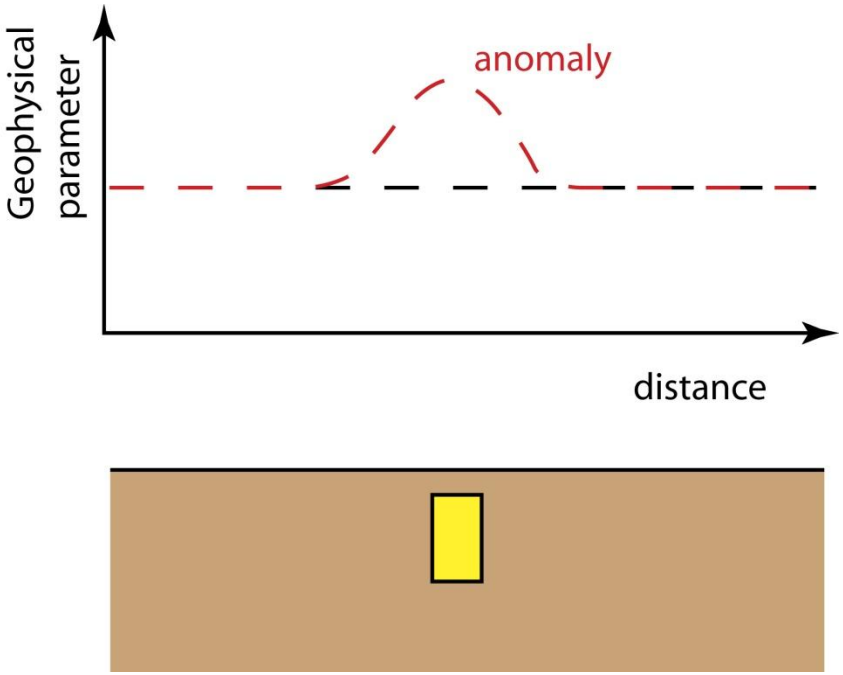
Definition: Contrast

To characterize different material using geophysics, a contrast must exist (i.e. a difference in the physical properties)

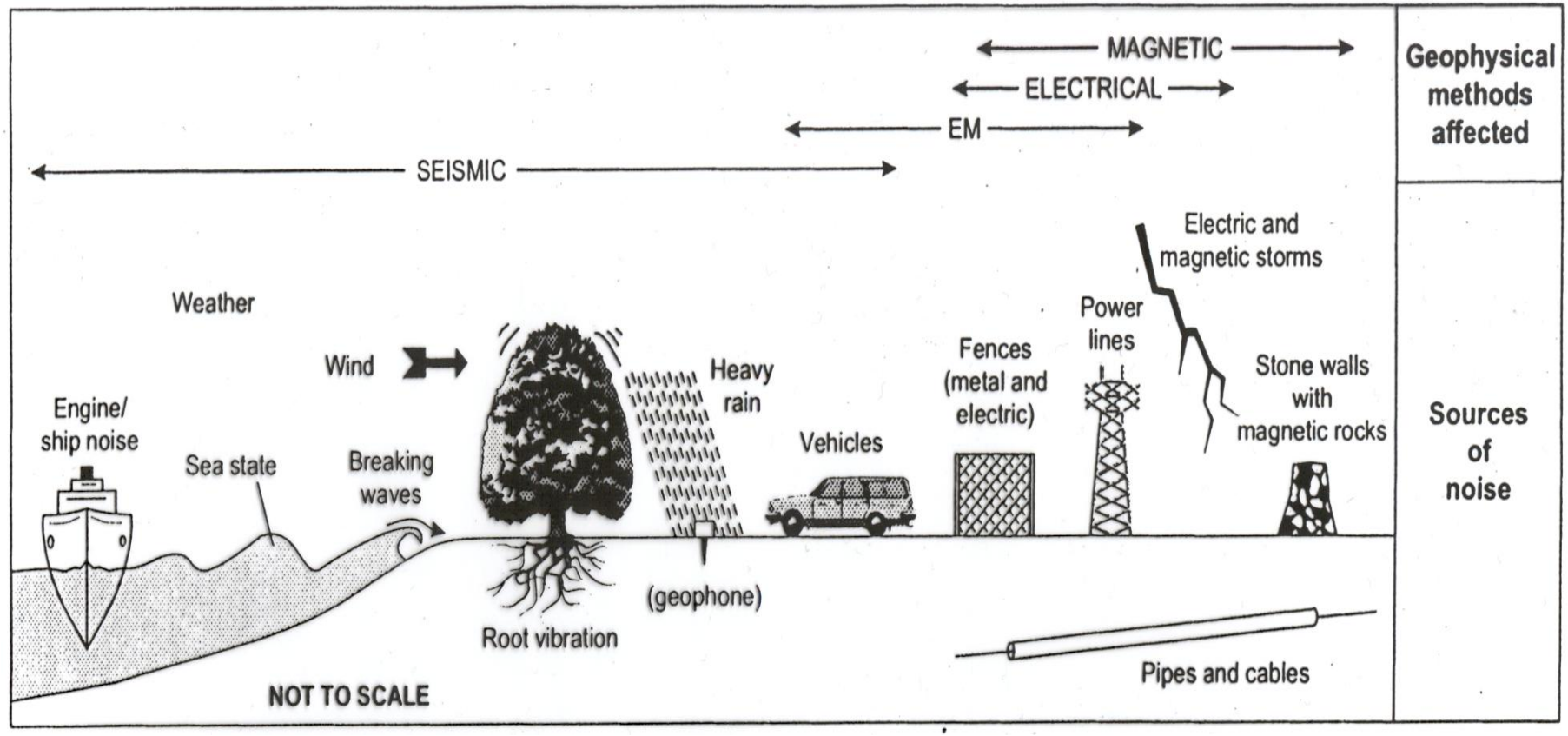


Definition: Anomaly

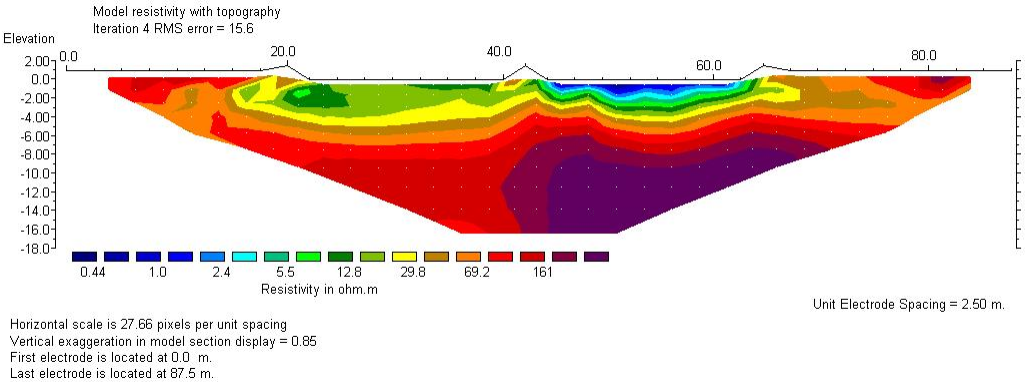


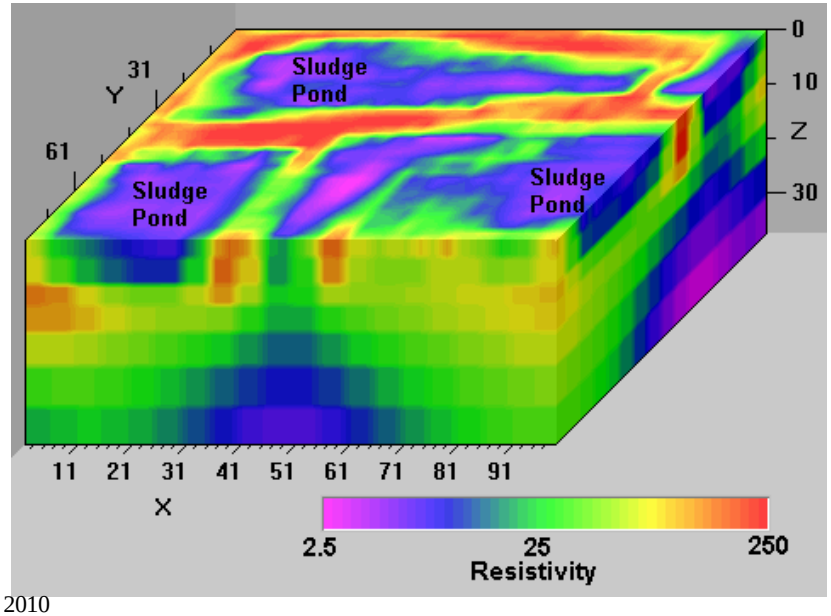


Noise in Geophysics

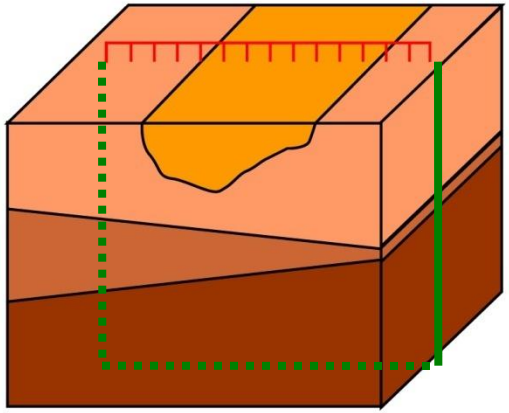


2D and 3D Models

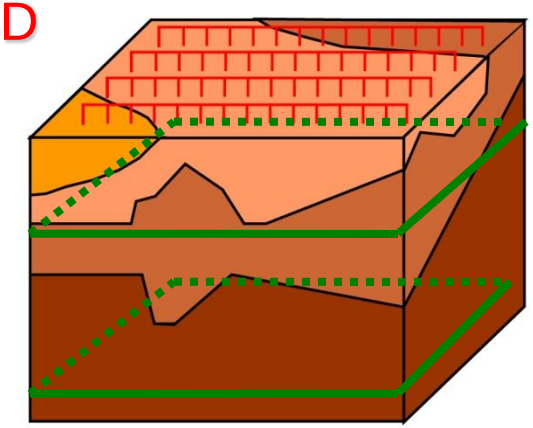




2D



3D



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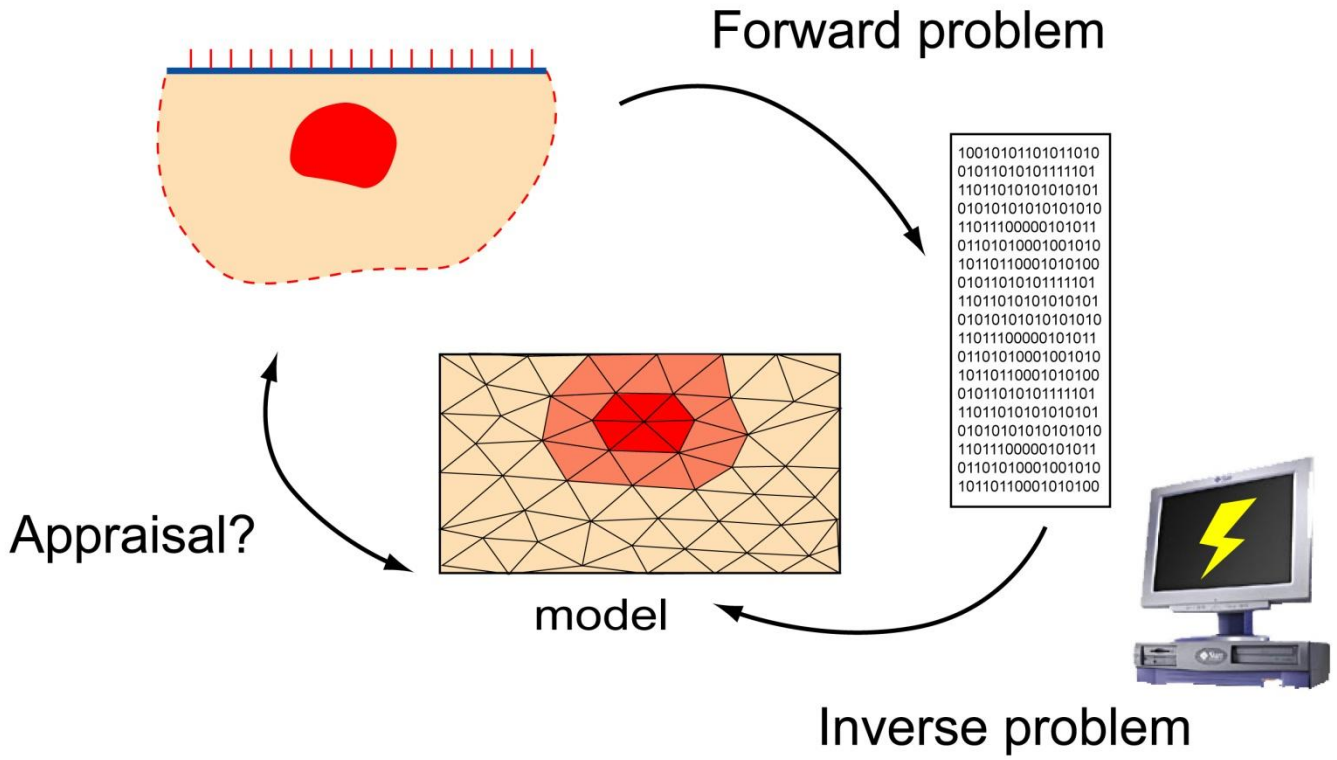
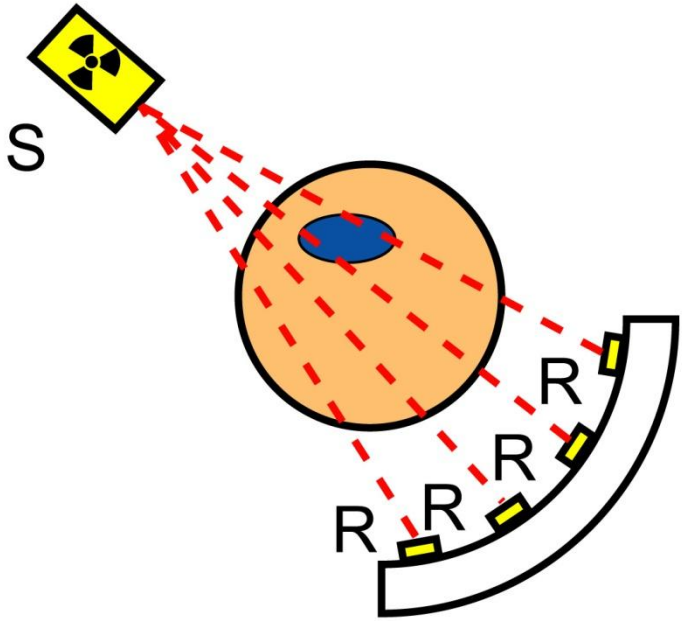
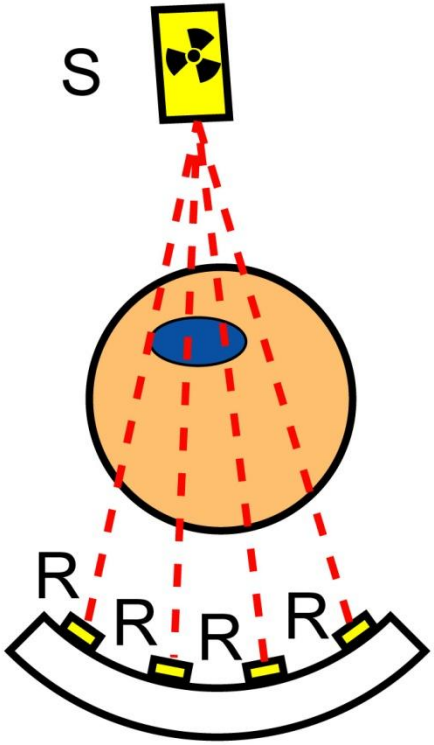
Forward and Inverse Problems



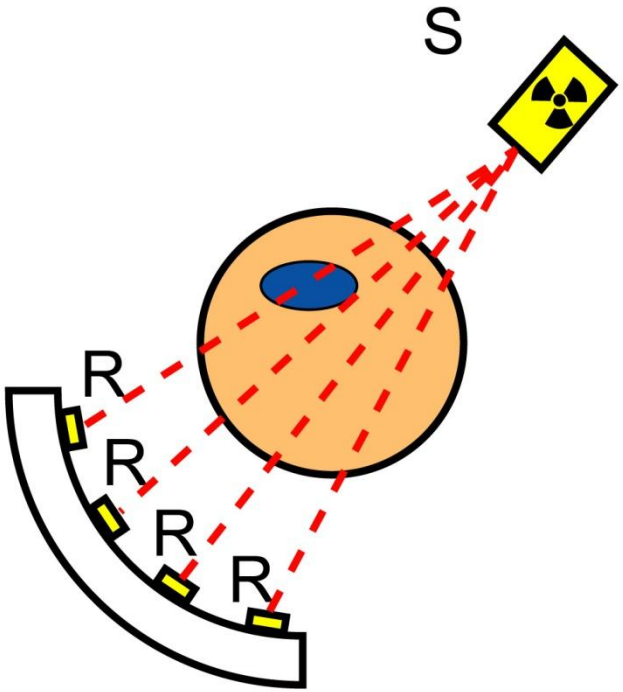
RX Tomography (CT-Scan)



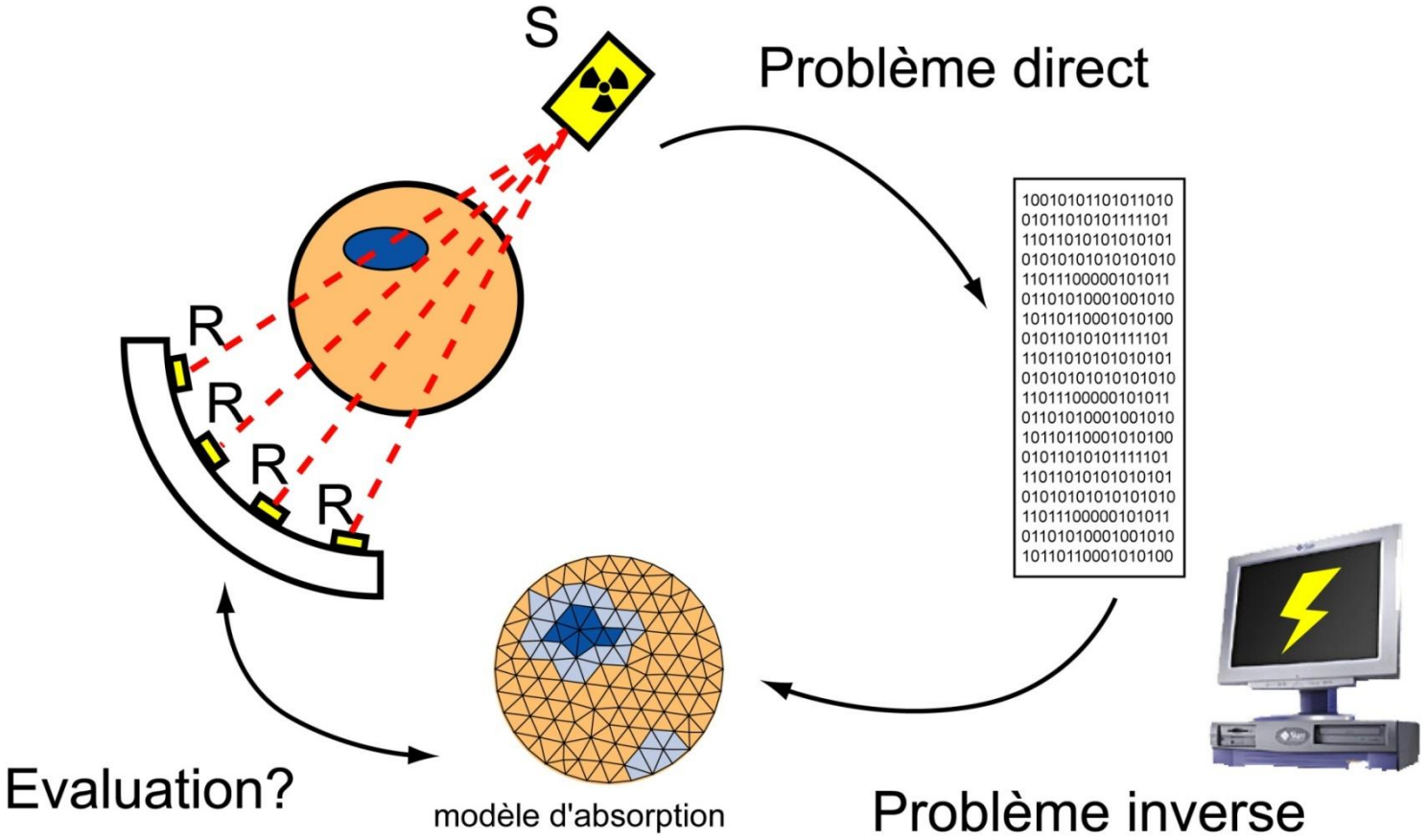
RX Tomography (CT-Scan)



RX Tomography (CT-Scan)



Forward and Inverse Problems



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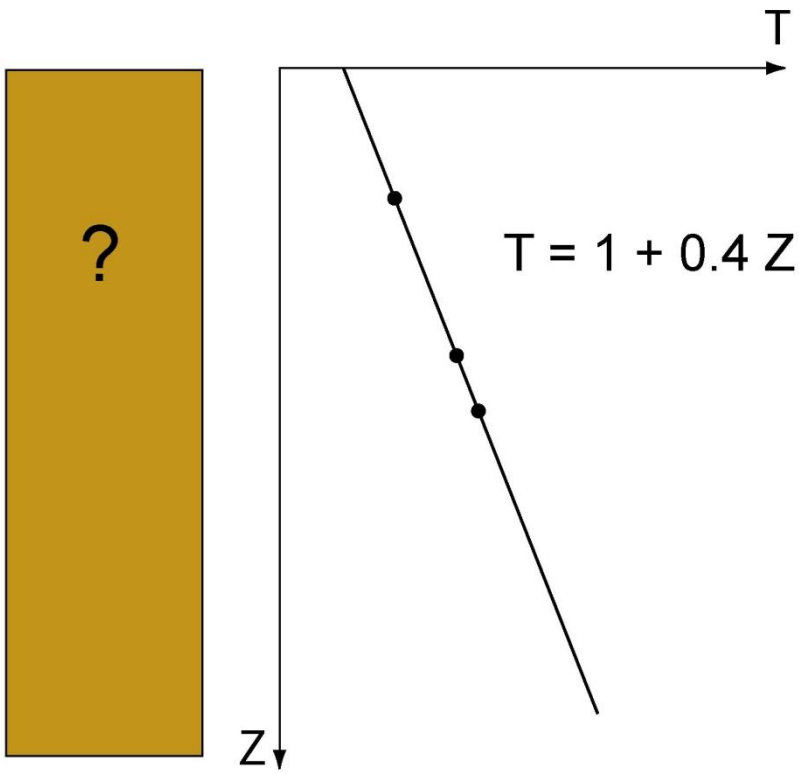
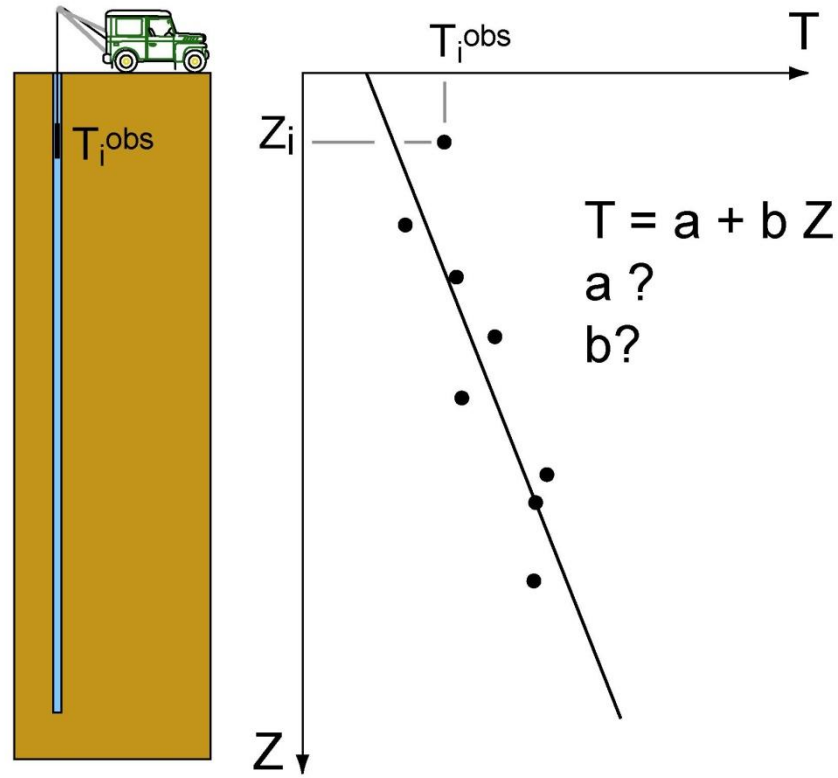
Direct and Inverse Problems



Direct Problem



Inverse Problem

Note: equation of a straight line: y=bx+a

Linear Inverse Problem:

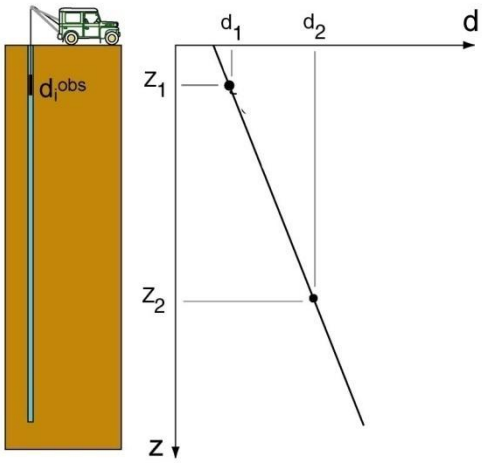
Even-determined Problem

$$T_1 = a + bZ_1$$
$$T_2 = a + bZ_2$$

$$\longrightarrow \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad \mathbf{d} = \mathbf{G}\,\mathbf{m}$$

$$\downarrow$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \end{bmatrix}^{-1} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \text{or} \quad \mathbf{m} = \mathbf{G}^{-1}\,\mathbf{d}$$



- Nb of data = Nb of model parameters
- Suppose noise-free data!

Linear Inverse Problem:

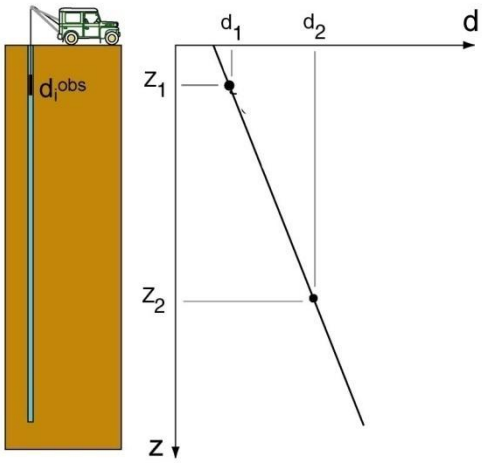
Even-determined Problem

$$T_1 = a + bZ_1 \qquad 19°C = a + b*2m$$
$$T_2 = a + bZ_2 \longrightarrow 22°C = a + b*8m \longrightarrow \begin{bmatrix} 19 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad \mathbf{d} = \mathbf{G}\,\mathbf{m}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \boxed{\frac{1}{6}\begin{bmatrix} 8 & -2 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} 19 \\ 22 \end{bmatrix} \quad \text{or} \quad \mathbf{m} = \boxed{\mathbf{G}^{-1}}\,\mathbf{d}$$

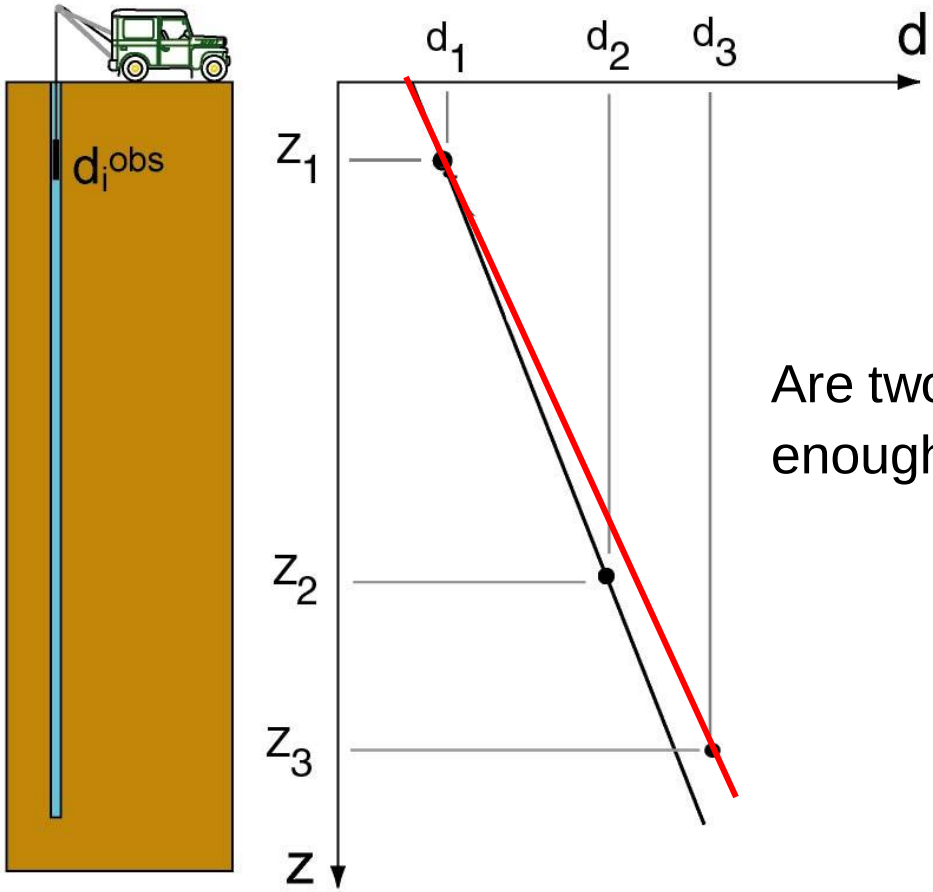
$a = 0.5$ (slope)

$b = 18°C$ (surface temperatu re)



Linear Inverse Problem:

Even-determined Problem



Are two data points enough?

Linear Inverse Problem

Over-determined Problem

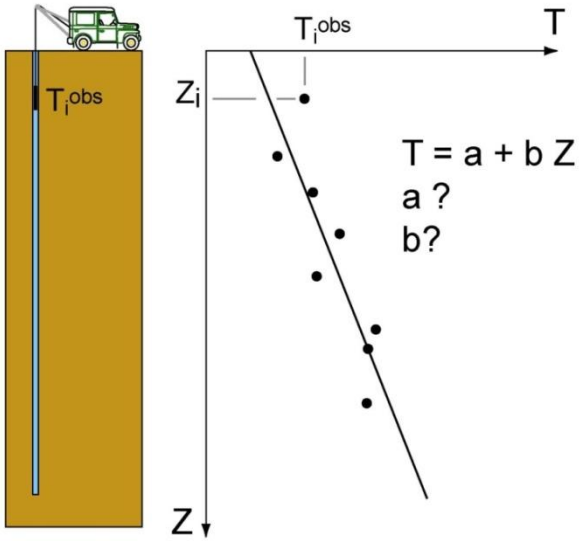
$$T_1 = a + bZ_1$$
$$T_2 = a + bZ_2$$
$$\vdots$$
$$T_N = a + bZ_N$$

$$\longrightarrow \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \\ \vdots & \vdots \\ 1 & Z_N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

or $\mathbf{d} = \mathbf{G}\ \mathbf{m}$

$$\downarrow$$

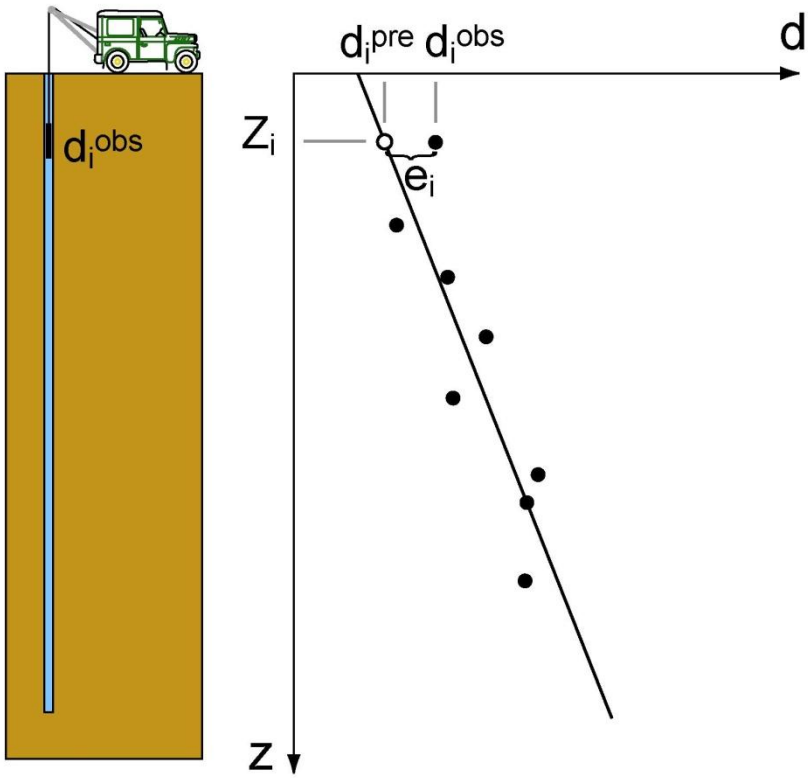
$$\mathbf{m} = \mathbf{G}^{-\mathbf{g}}\ \mathbf{d}$$



- Nb of data > Nb of model parameters
- G is no longer a square matrix: cannot be inverted!

Linear Inverse Problem

Over-determined Problem



$$L_2\ norm: \quad \|\mathbf{e}\|_2 = \left[\sum_i |e_i|^2\right]^{\frac{1}{2}}$$

$$\text{Min}\ \ e_i = d_i^{obs} - d_i^{pre}$$

$$E = \mathbf{e}^{\mathbf{T}}\,\mathbf{e} = (\mathbf{d} - \mathbf{Gm})^{\mathbf{T}}\,(\mathbf{d} - \mathbf{Gm})$$

$$\downarrow$$

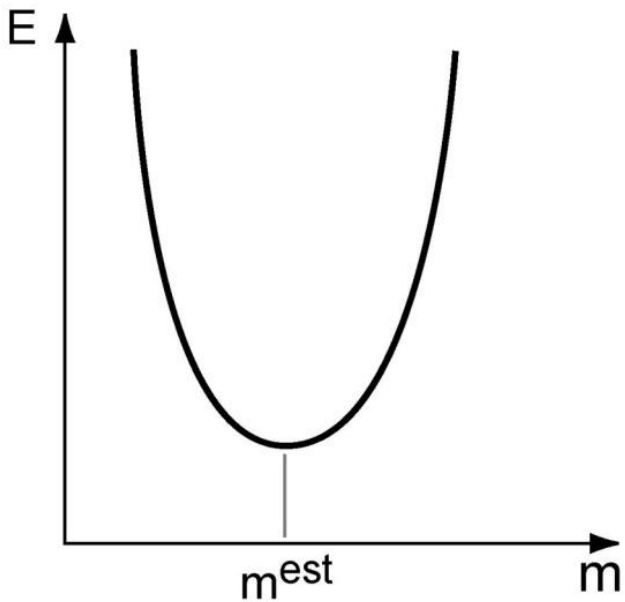
$$\frac{\partial E}{\partial m} = 0$$

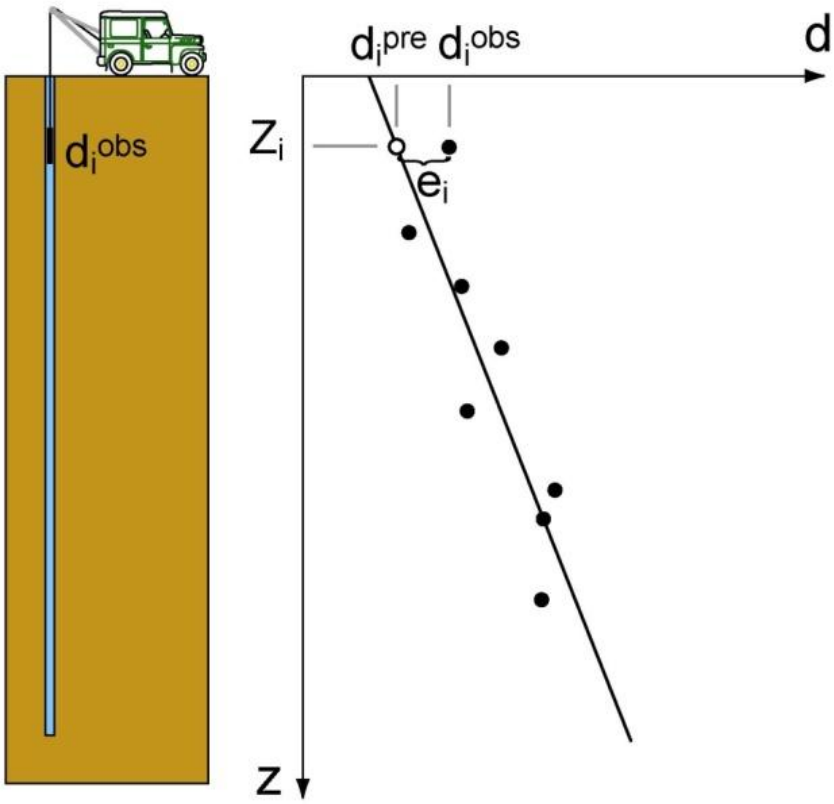
$$\downarrow$$

Solution of inverse problem: $\mathbf{m}^{\mathbf{est}} = \boxed{\mathbf{G}^{-\mathbf{g}}}\,\mathbf{d}$ or $\mathbf{m}^{\mathbf{est}} = \boxed{[\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\,\mathbf{G}^{\mathbf{T}}}\,\mathbf{d}$

Linear Inverse Problem

Over-determined Problem





$$E = \mathbf{e}^{\mathbf{T}}\,\mathbf{e}$$

$$\frac{\partial E}{\partial m} = 0$$

Linear Inverse Problem

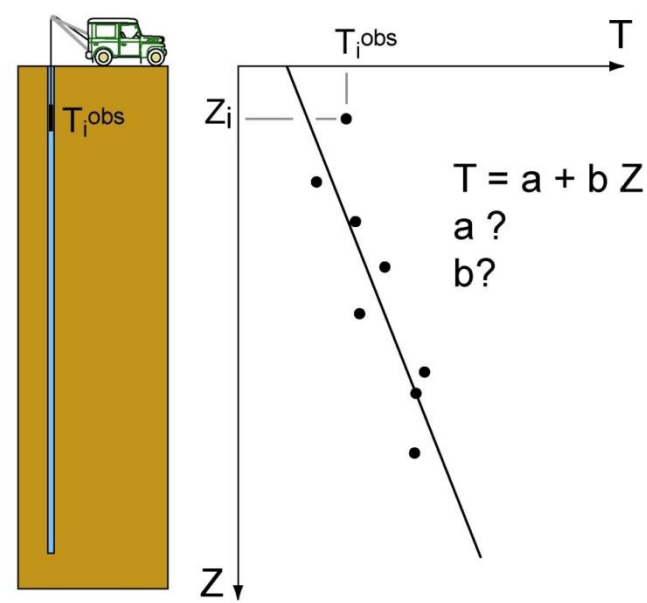
Over-determined Problem

$$T_1 = a + bZ_1$$
$$T_2 = a + bZ_2$$
$$\vdots$$
$$T_N = a + bZ_N$$

$$\longrightarrow \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \\ \vdots & \vdots \\ 1 & Z_N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad \mathbf{d} = \mathbf{G}\,\mathbf{m}$$

$$\downarrow$$

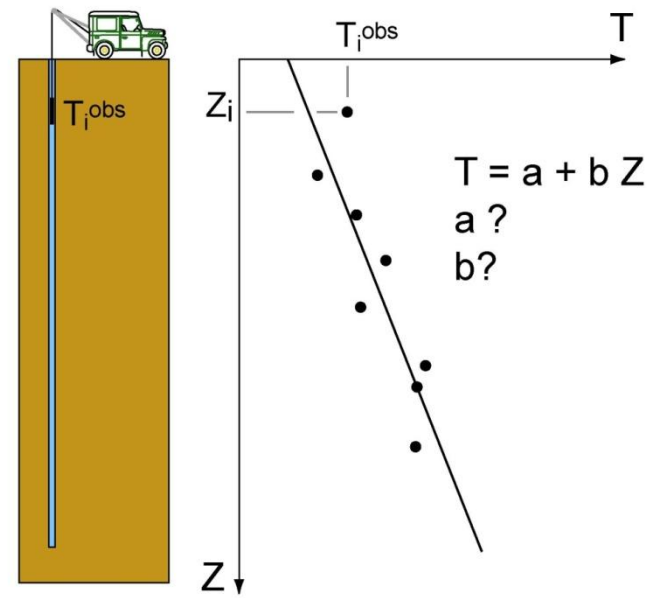
$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$



Linear Inverse Problem

Over-determined Problem

$$\mathbf{m}^{\mathbf{est}} = \boxed{[\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\,\mathbf{G}^{\mathbf{T}}}\,\mathbf{d}$$

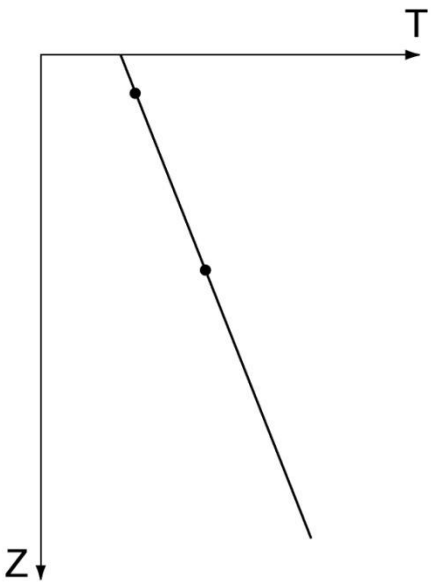


$$\begin{bmatrix} a \\ b \end{bmatrix} = \boxed{\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ Z_1 & Z_2 & \cdots & Z_N \end{bmatrix} \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \\ \vdots & \vdots \\ 1 & Z_N \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ Z_1 & Z_2 & \cdots & Z_N \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}$$

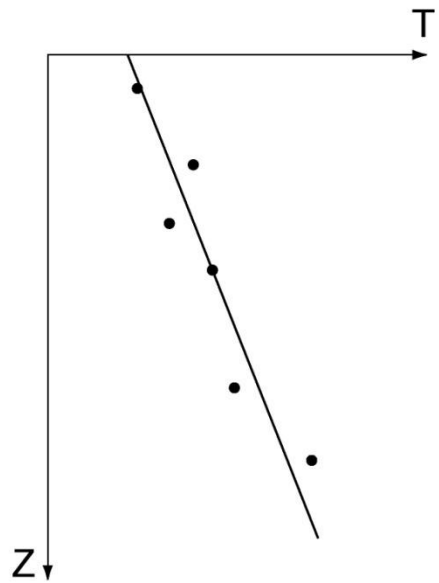
Linear Inverse Problem

Effect of Number of Data

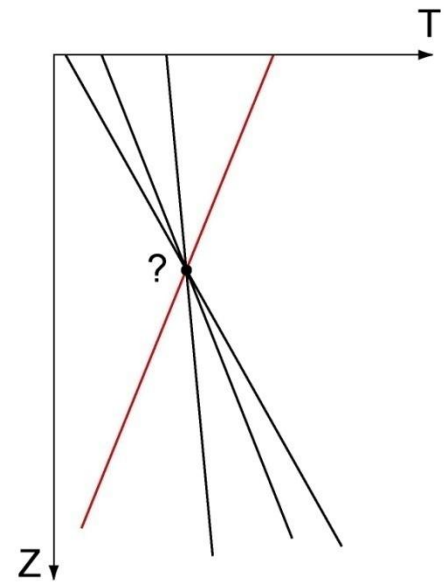
Even-determined



Over-determined



Under-determined

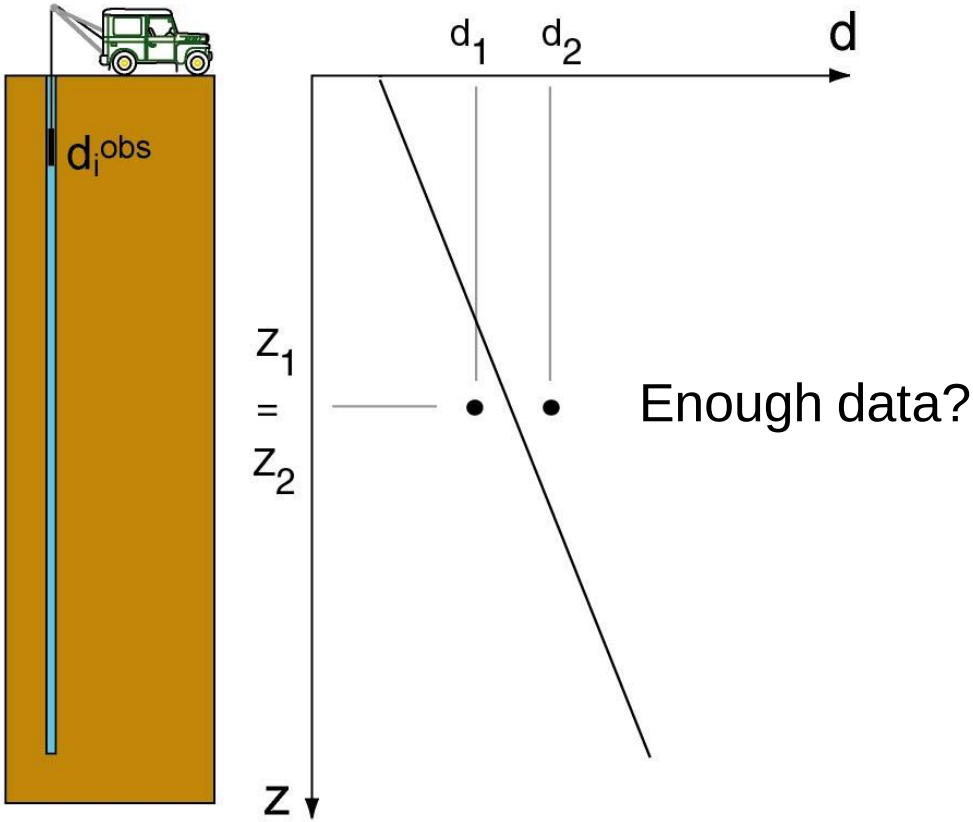


$$\mathbf{m} = \mathbf{G}^{-1}\ \mathbf{d}$$

$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\ \mathbf{G}^{\mathbf{T}}\ \mathbf{d}$$

Linear Inverse Problem

A Mixed Problem



Linear Inverse Problem

A Mixed Problem: Issue

$$T_1 = a + bZ$$
$$T_2 = a + bZ$$

$$\longrightarrow \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 1 & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad \mathbf{d} = \mathbf{G}\,\mathbf{m}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \boxed{\left(\begin{bmatrix} 1 & 1 \\ Z & Z \end{bmatrix} \begin{bmatrix} 1 & Z \\ 1 & Z \end{bmatrix} \right)^{-1}} \begin{bmatrix} 1 & 1 \\ Z & Z \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \text{or} \quad \mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$

$$\Downarrow$$

$$\boxed{\frac{1}{2(Z^2 - Z^2)} \begin{bmatrix} Z^2 & -Z \\ -Z & 1 \end{bmatrix}} \Rightarrow \boxed{\frac{1}{0} \begin{bmatrix} Z^2 & -Z \\ -Z & 1 \end{bmatrix}}$$

Linear Inverse Problem

A Mixed Problem: Solution

$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d} \longrightarrow \mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$

Least-squares inversion with Marquardt-Levenberg modification

$$\begin{bmatrix} a \\ b \end{bmatrix} = \boxed{\left(\begin{bmatrix} 1 & 1 \\ Z & Z \end{bmatrix}\begin{bmatrix} 1 & Z \\ 1 & Z \end{bmatrix} + \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}\right)^{-1}} \begin{bmatrix} 1 & 1 \\ Z & Z \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\frac{1}{(2\varepsilon + 2Z^2\varepsilon + \varepsilon^2)}\begin{bmatrix} 2Z^2 + \varepsilon & -2Z \\ -2Z & 2 + \varepsilon \end{bmatrix}}$$

ε small, e.g. 0.0001

Linear Inverse Problem

Including a priori Information

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} 1 & Z_1 \\ 1 & Z_2 \end{bmatrix} + \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 \\ Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

or

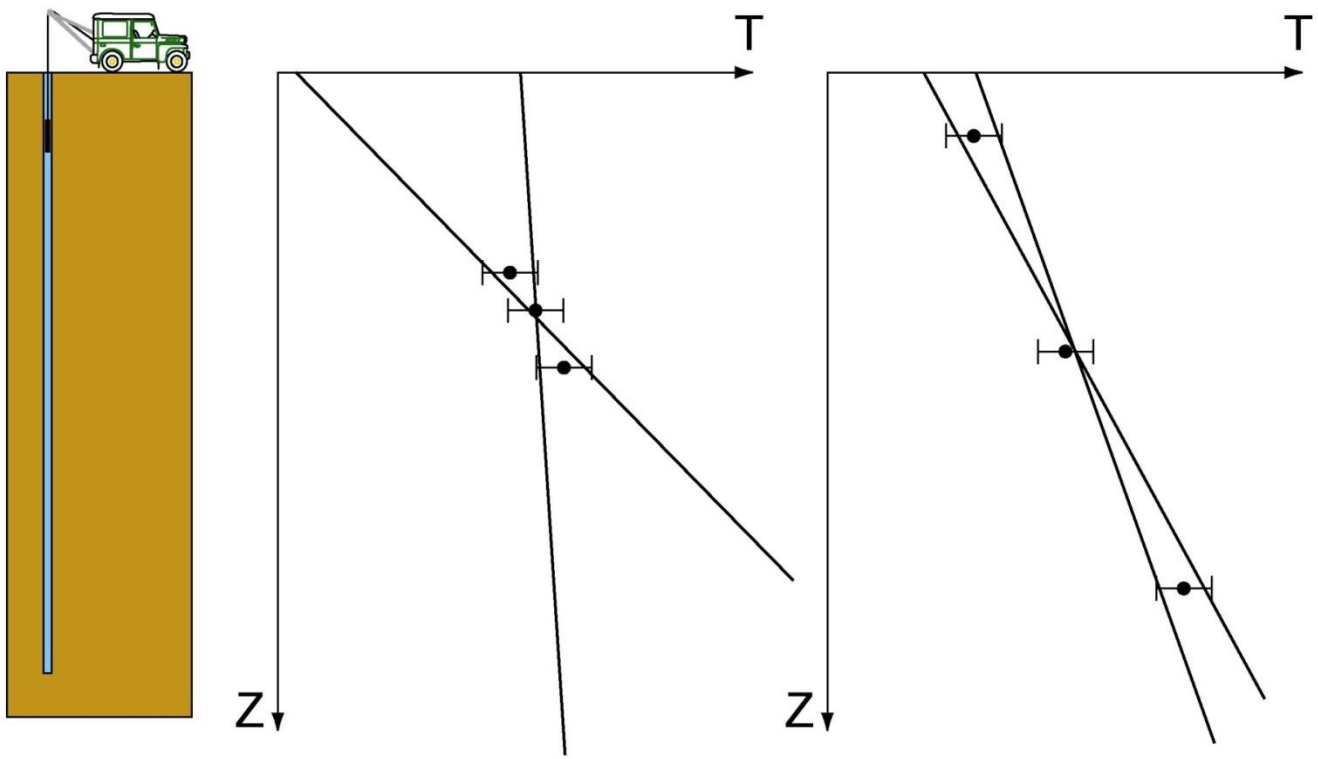
$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$

For example, add a priori information on the surface temperature (e.g. *b* = 18°C)
In this case, the solution for *a* (slope) should reflect:

- information on the measured data
- a priori information on *b*

Linear Inverse Problem

Effect of Sampling on Solution Variance



Because of measurement errors, inversion solution is NOT UNIQUE!

Linear Inverse Problem

Including a priori Information

Mixed (Marquardt-Levenberg)

$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\ \mathbf{W}_{\mathbf{e}}\ \mathbf{G} + \boldsymbol{\varepsilon}\ \mathbf{W}_{\mathbf{m}}]^{-1}\ \mathbf{G}^{\mathbf{T}}\ \mathbf{W}_{\mathbf{e}}\ \mathbf{d}$$

W are weighting matrices

More information in this paper:

Un algorithme d'inversion par moindres carrés pondérés: application aux données géophysiques par méthodes électromagnétiques en domaine fréquence: Marescot, 2003, Bull. Soc. vaud. Sc. nat. 88.3: 277-300. www. tomoquest.com

Linear Inverse Problem

Key Concepts

- Direct vs. Inverse problems
- A priori Information
- **G** can be pre-evaluated
- Even-, Over-, Under-determined inverse problems
- Mixed-determined inverse problem

$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$

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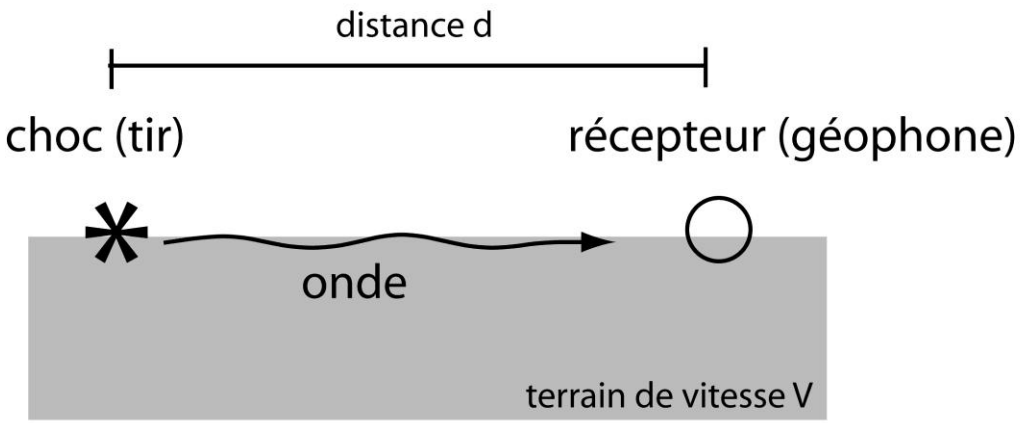
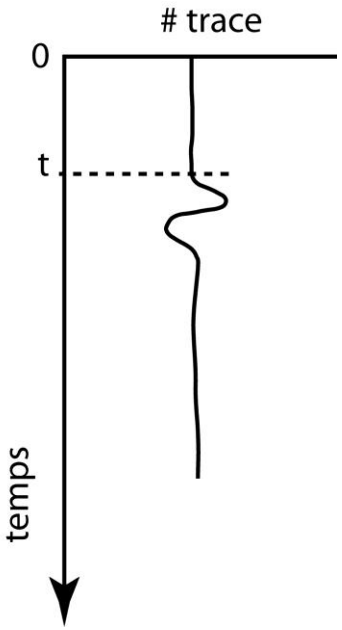
Non-linear Inverse Problem

Seismic Measurement Principle

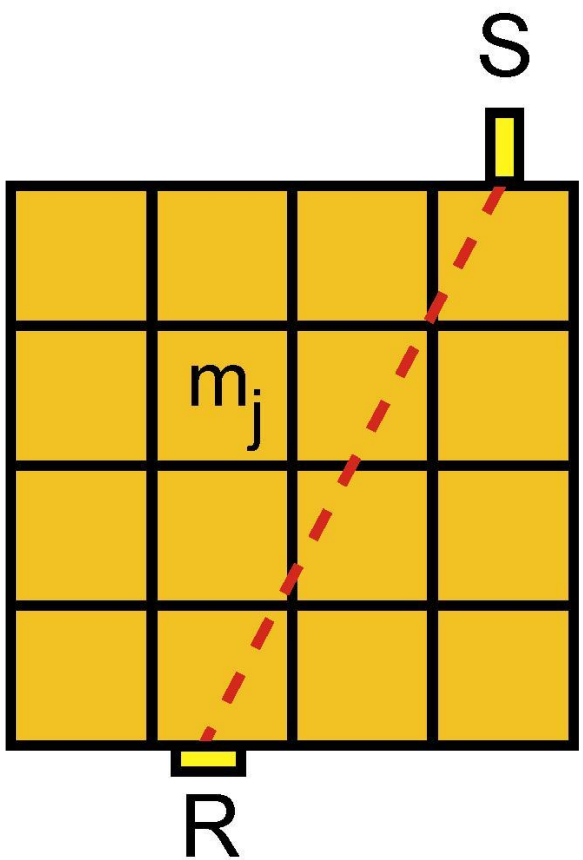


$$V = d / t$$

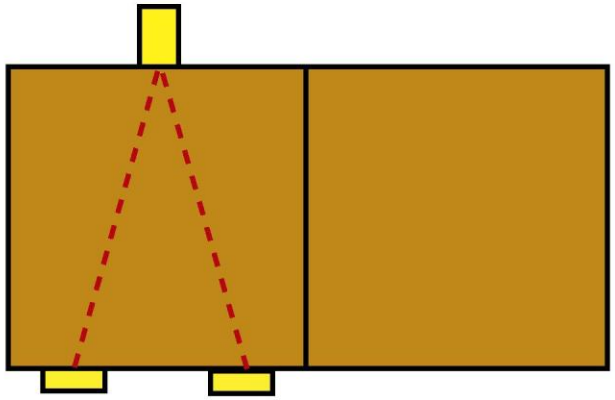


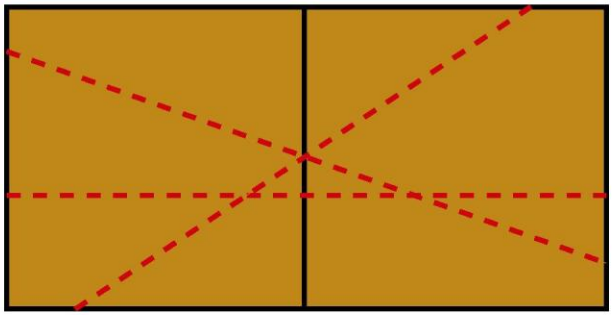
Non-linear Inverse Problem

Discrete Problem



A Mixed Problem…

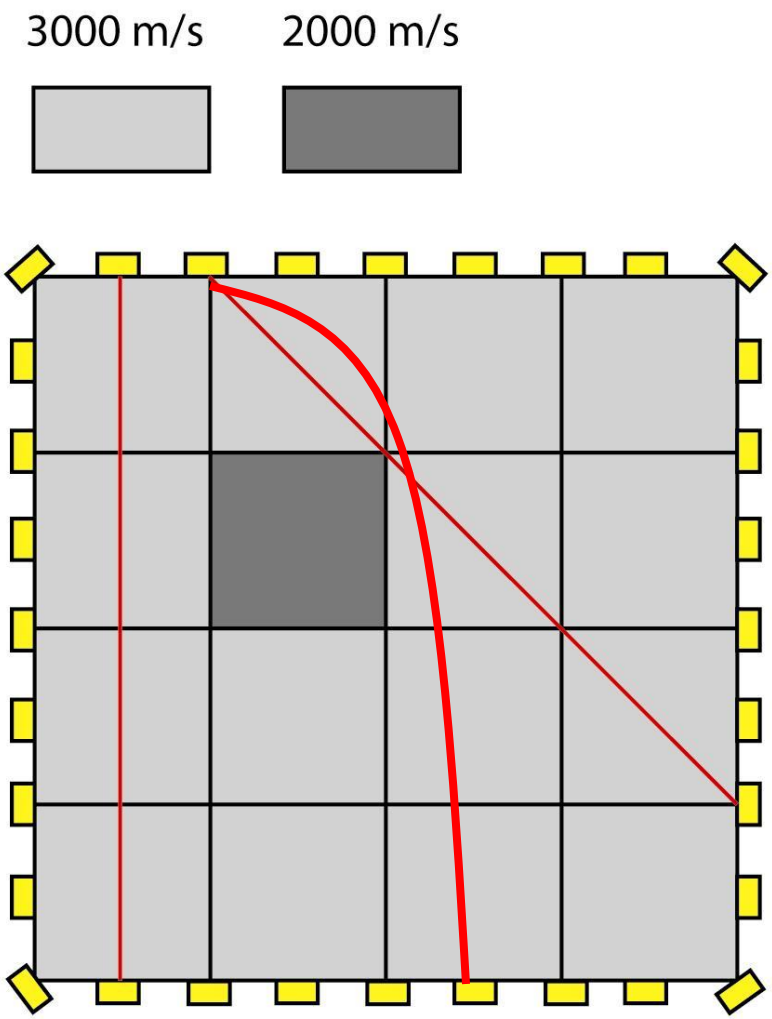




Linear / Non-linear Problems

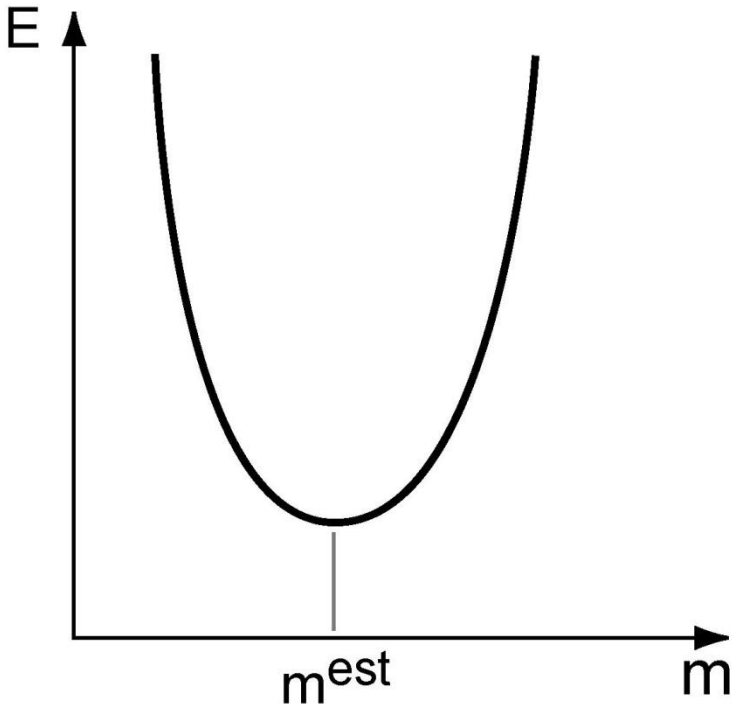
In fact, seismic tomography is NON-LINEAR, since the ray paths depend on the UNKNOWN velocities in the model:

- No straight-line rays
- Iterative solution required

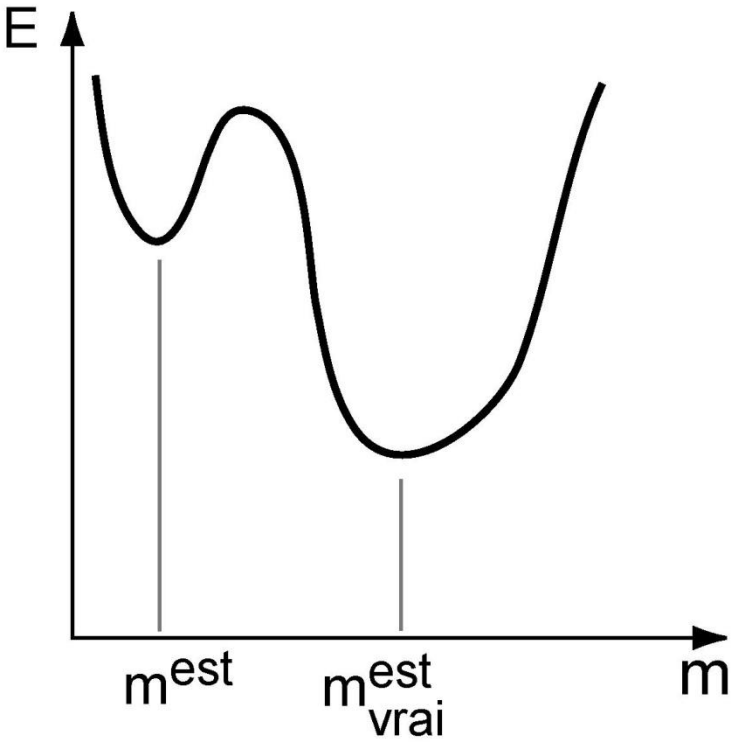


Non-linear Inverse Problem

Optimal Solution



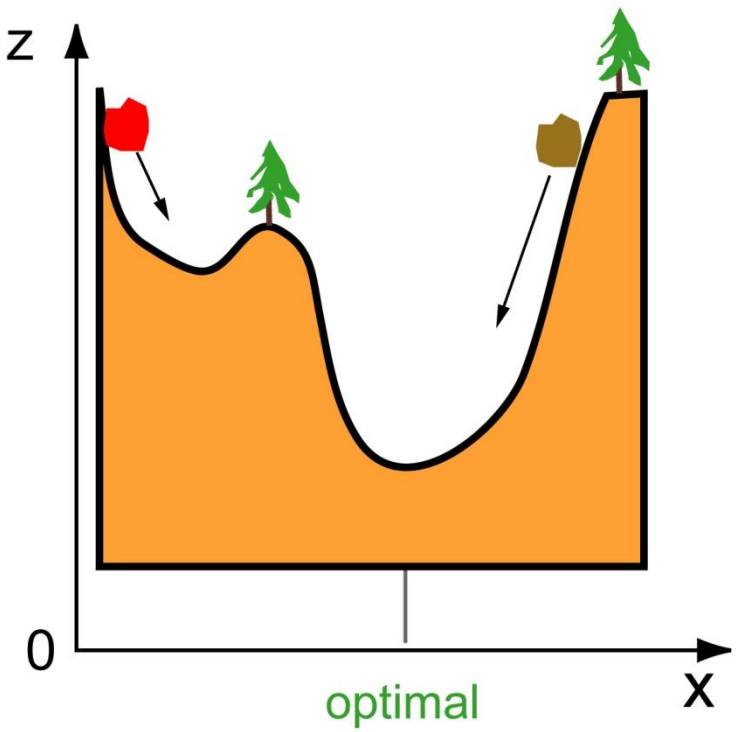
Linear Problem

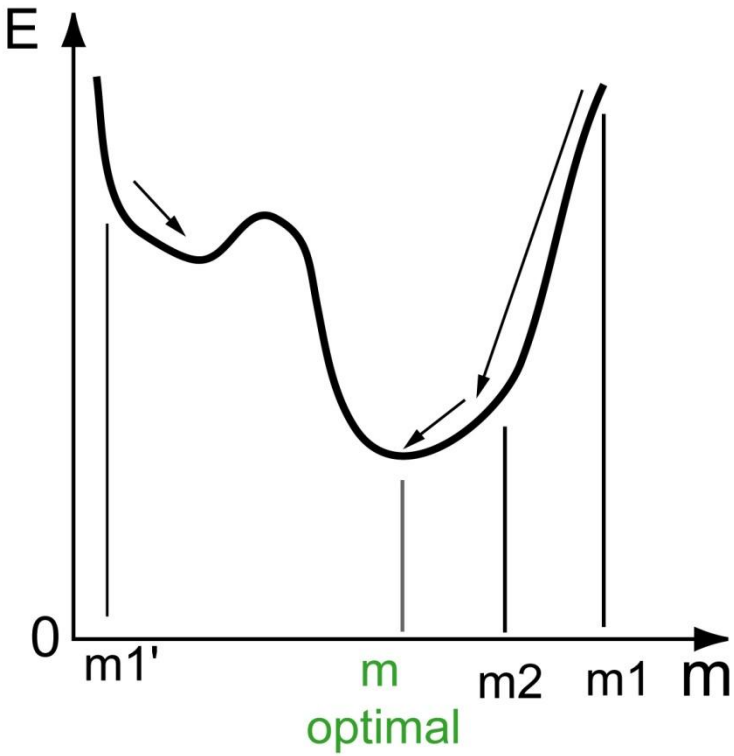


Non-linear Problem

Non-linear Inverse Problem

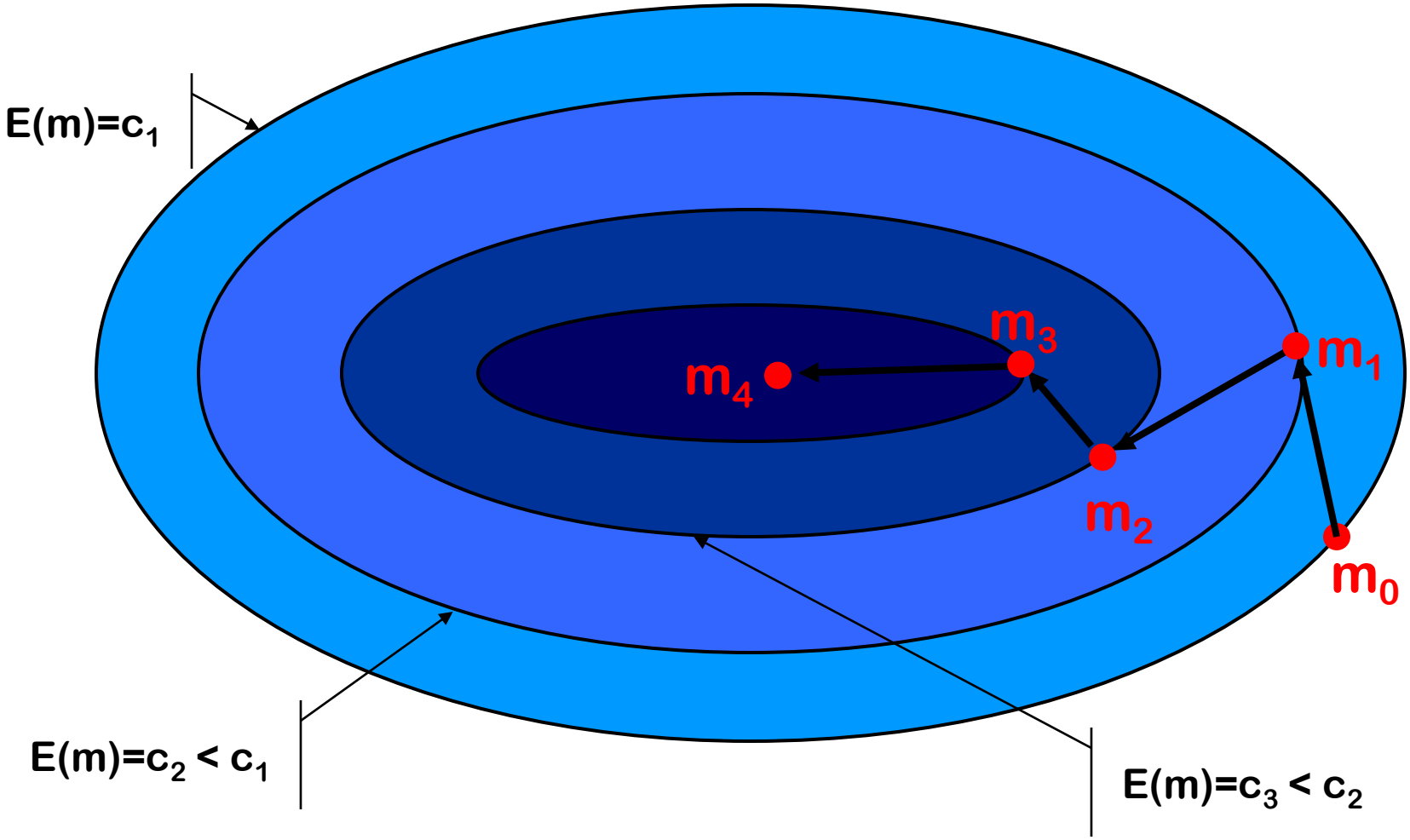
Descent Methods





Non-linear Inverse Problem

Descent Methods



Non-linear Inversion

Make the problem linear using a Taylor series around an estimated solution:

$$\mathbf{g(m)} \cong \mathbf{g(m_n^{est})} + \nabla \mathbf{g(m - m_n^{est})} = \mathbf{g(m_n^{est})} + \mathbf{G_n(m - m_n^{est})}$$

$$\downarrow$$

$$\mathbf{G_n \Delta m_{n+1}} = \mathbf{d} - \mathbf{g(m_n^{est})}$$

$$\mathbf{\Delta d} = \mathbf{d^{obs}} - \mathbf{d^{calc}} = \mathbf{G \Delta m} \qquad G_{ij} = \frac{\partial g(m)_i}{\partial m_j}$$

Non-linear Inversion

$$\Delta\mathbf{d} = \mathbf{d}^{\mathbf{obs}} - \mathbf{d}^{\mathbf{calc}} = \mathbf{G}\Delta\mathbf{m}$$

$$\downarrow$$

$$\Delta\mathbf{m} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\Delta\mathbf{d}$$

$$G_{ij} = \frac{\partial g(m)_i}{\partial m_j}$$

Compare with the linear inverse solution:

$$\mathbf{d} = \mathbf{G}\,\mathbf{m}$$

$$\mathbf{m}^{\mathbf{est}} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\mathbf{d}$$

Non-linear Inversion

1) Donner une valeur initiale pour le modèle $\mathbf{m}$

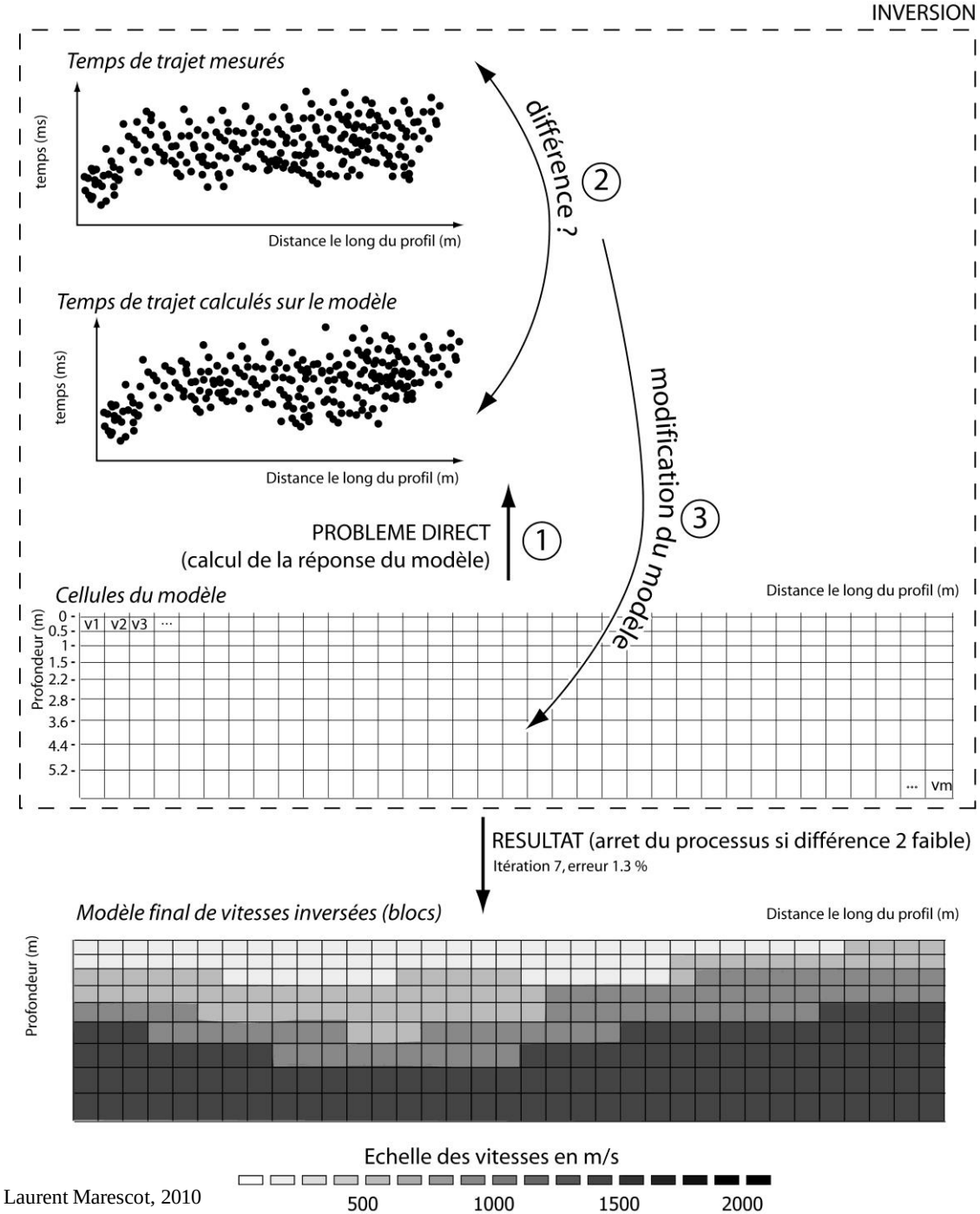
2) Calculer la réponse de ce modèle ($\mathbf{d}^{\mathbf{calc}}$). C'est le problème direct.

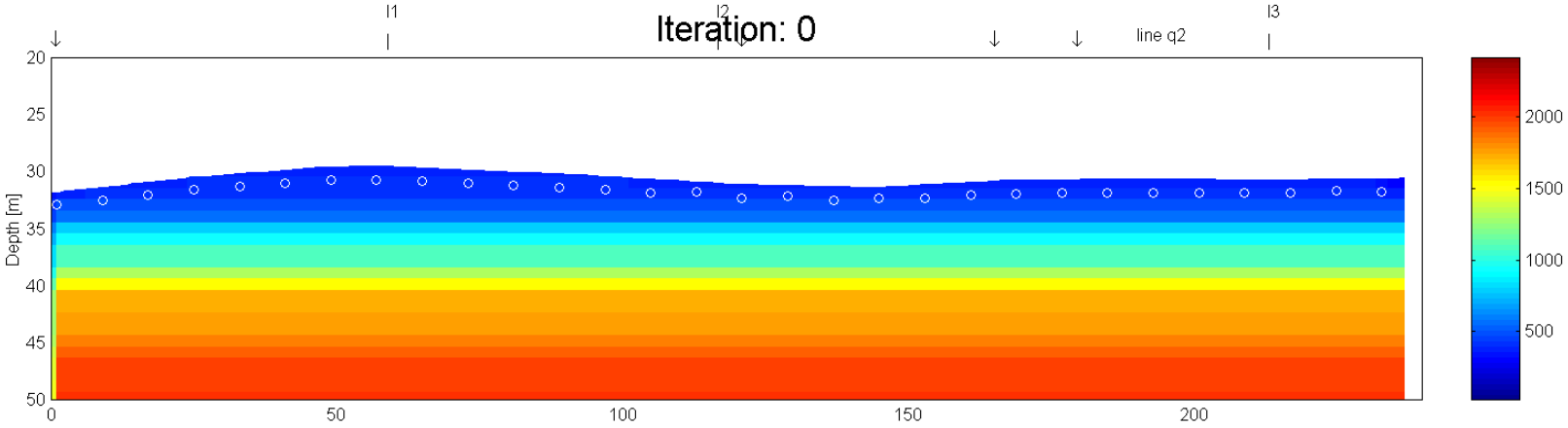
3) Evaluer la correction $\Delta\mathbf{m}$ à apporter au modèle et corriger le modèle

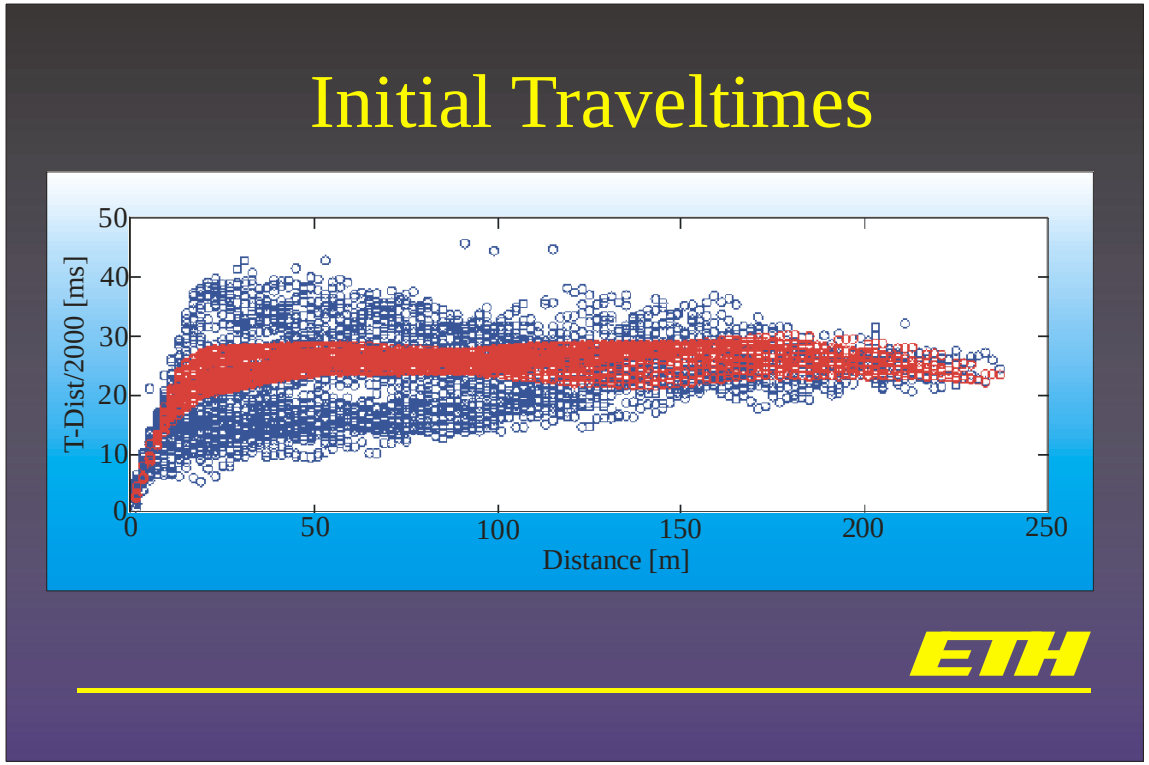
$$\Delta\mathbf{m} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\Delta\mathbf{d}$$

4) Recommencer le processus au point 2) jusqu'à convergence

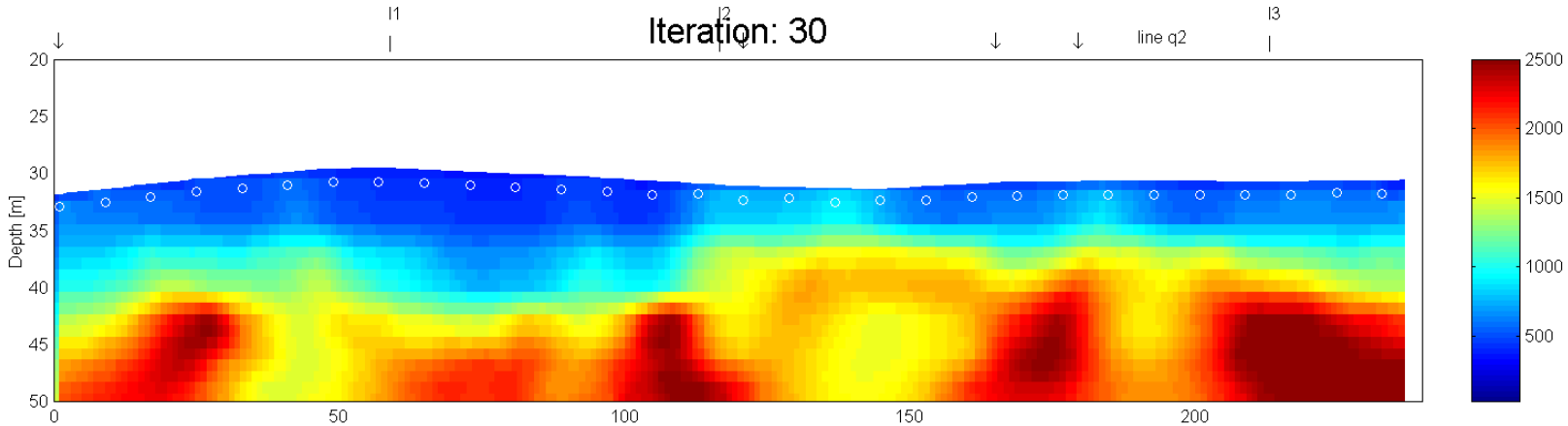
Seismic tomography inversion

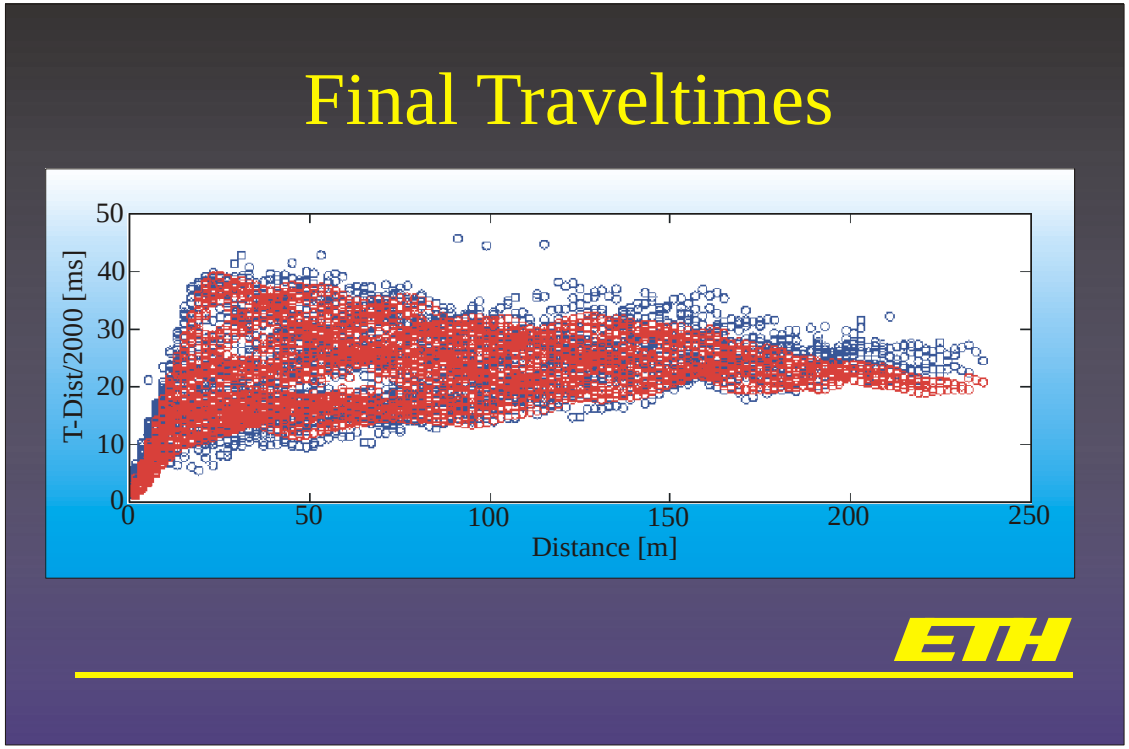



Iteration: 0
l1
l2
l3
line q2
Depth [m]
20
25
30
35
40
45
50
0
50
100
150
200
2000
1500
1000
500


Initial Traveltimes
T-Dist/2000 [ms]
50
40
30
20
10
0
0
50
100
150
200
250
Distance [m]
ETH

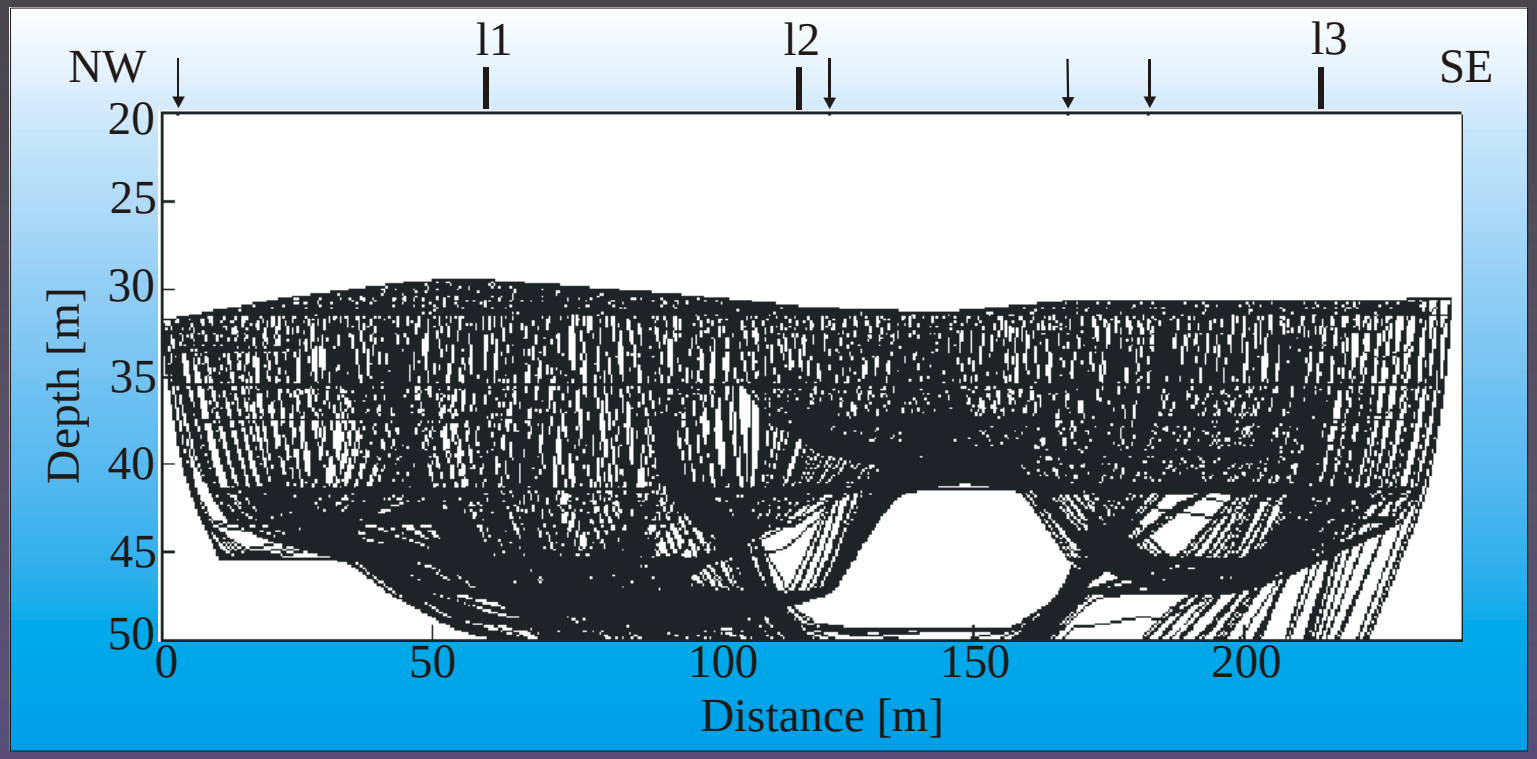
Source: ETHZ


Iteration: 30
l1
l2
l3
line q2
Depth [m]
20
25
30
35
40
45
50
0
50
100
150
200
2500
2000
1500
1000
500


Final Traveltimes
T-Dist/2000 [ms]
50
40
30
20
10
0
0
50
100
150
200
250
Distance [m]
ETH

Source: ETHZ

Raypaths q2

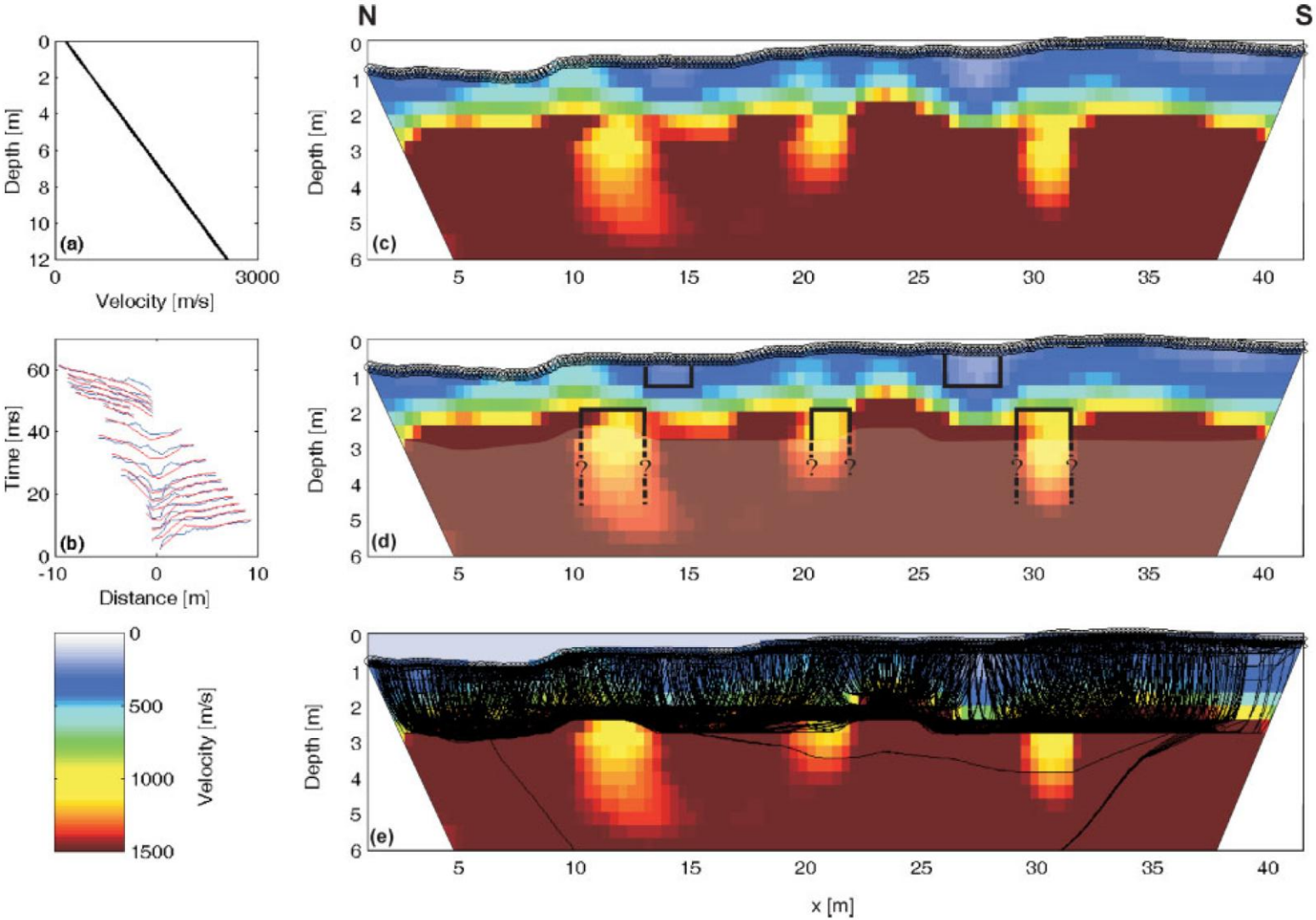


Source: ETHZ

Combined Seismic Tomographic and Ultrashallow Seismic Reflection Study of an Early Dynastic Mastaba, Saqqara, Egypt

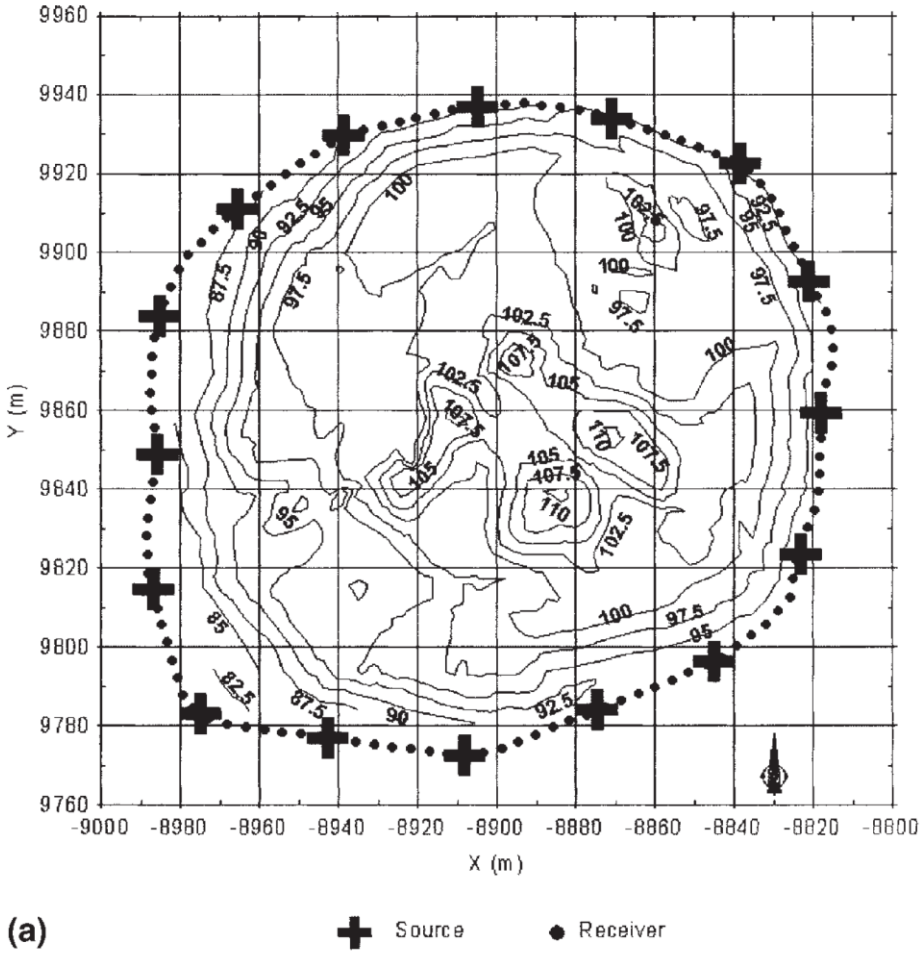
Metwaly et al., 2005,
Archeological Prospection, 12, 245-256

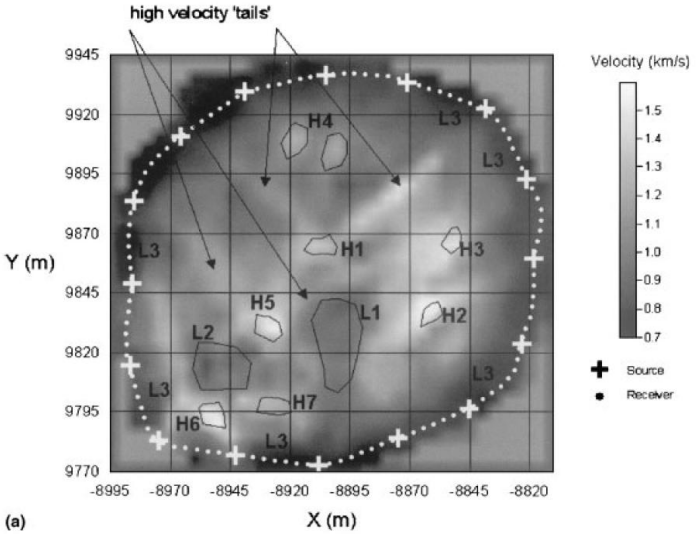
Refraction

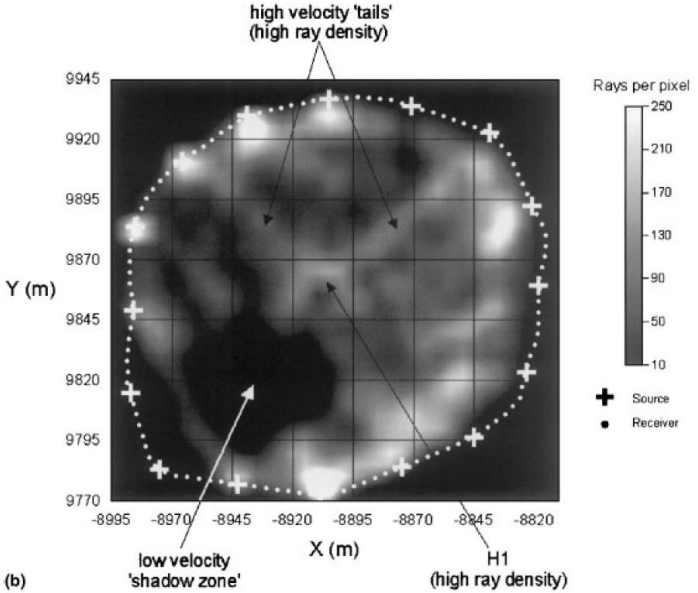


Metwaly et al., 2005, Archeological Prospection, 12, 245-256

Investigation of a Monumental Macedonian Tumulus by Three dimensional Seismic Tomography



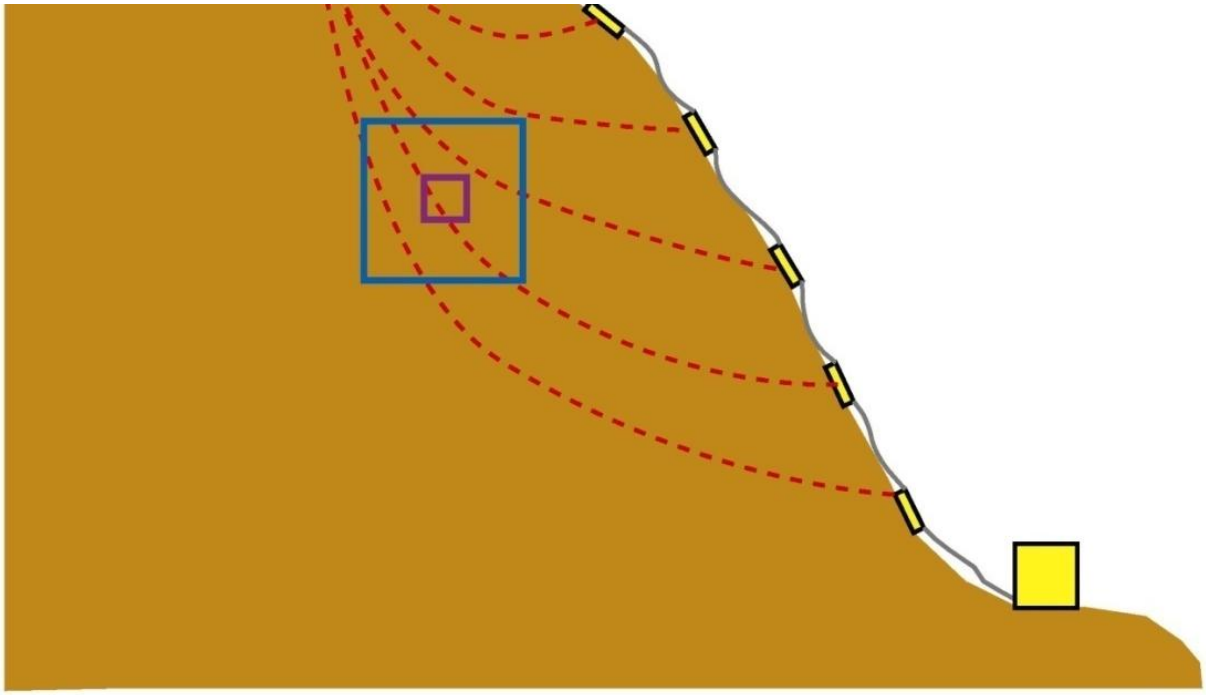




Polymenakos et al., 2004, Archeological Prospection, 11, 145-158

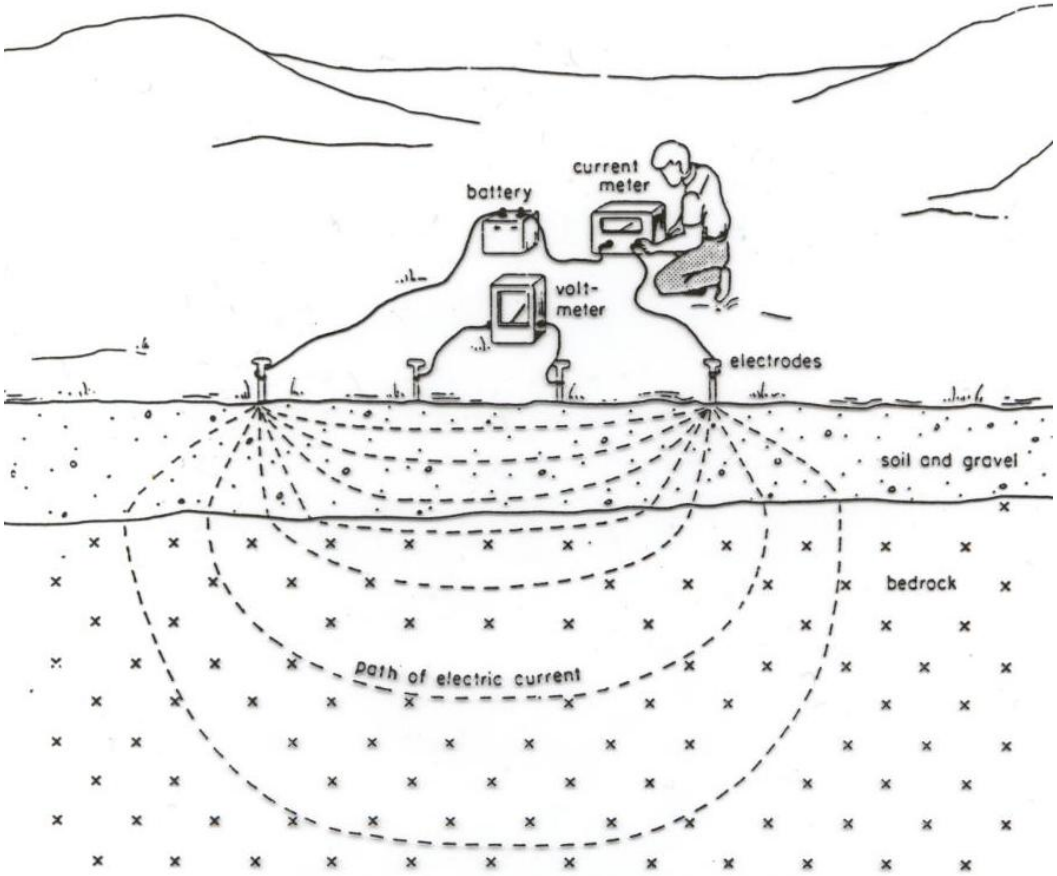
Non-linear Inverse Problem

Error vs Resolution



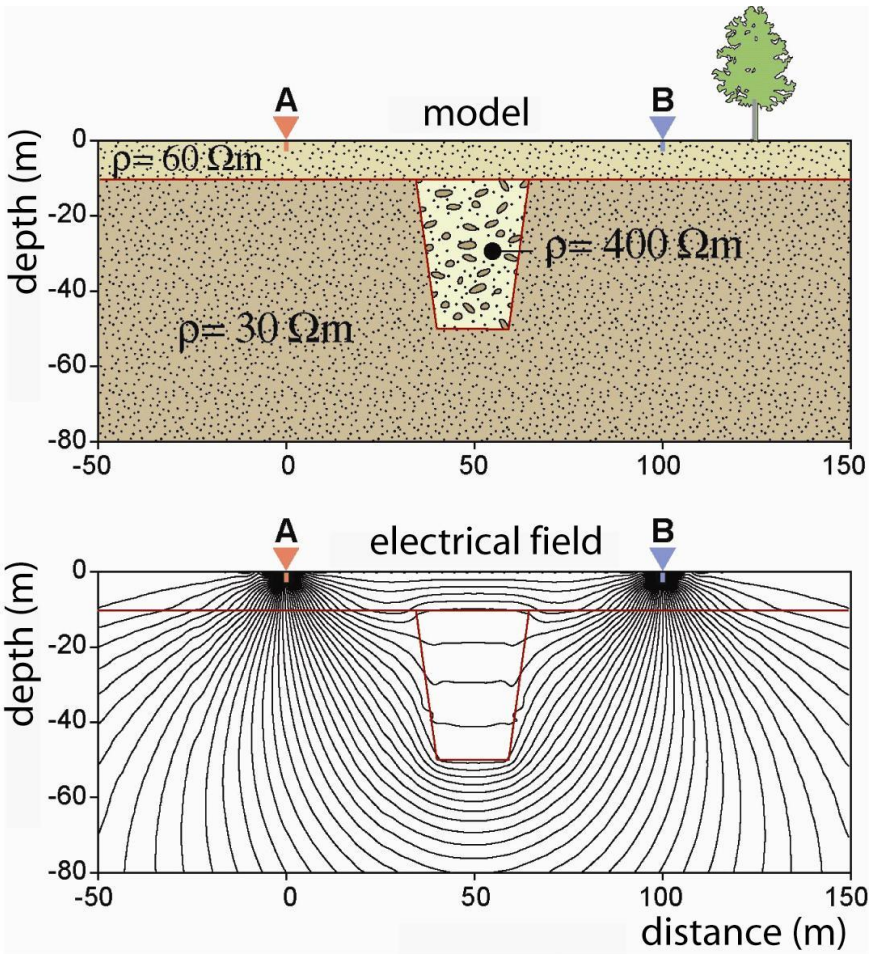
Non-linear Inverse Problem

Geoelectrical Measurement Principle

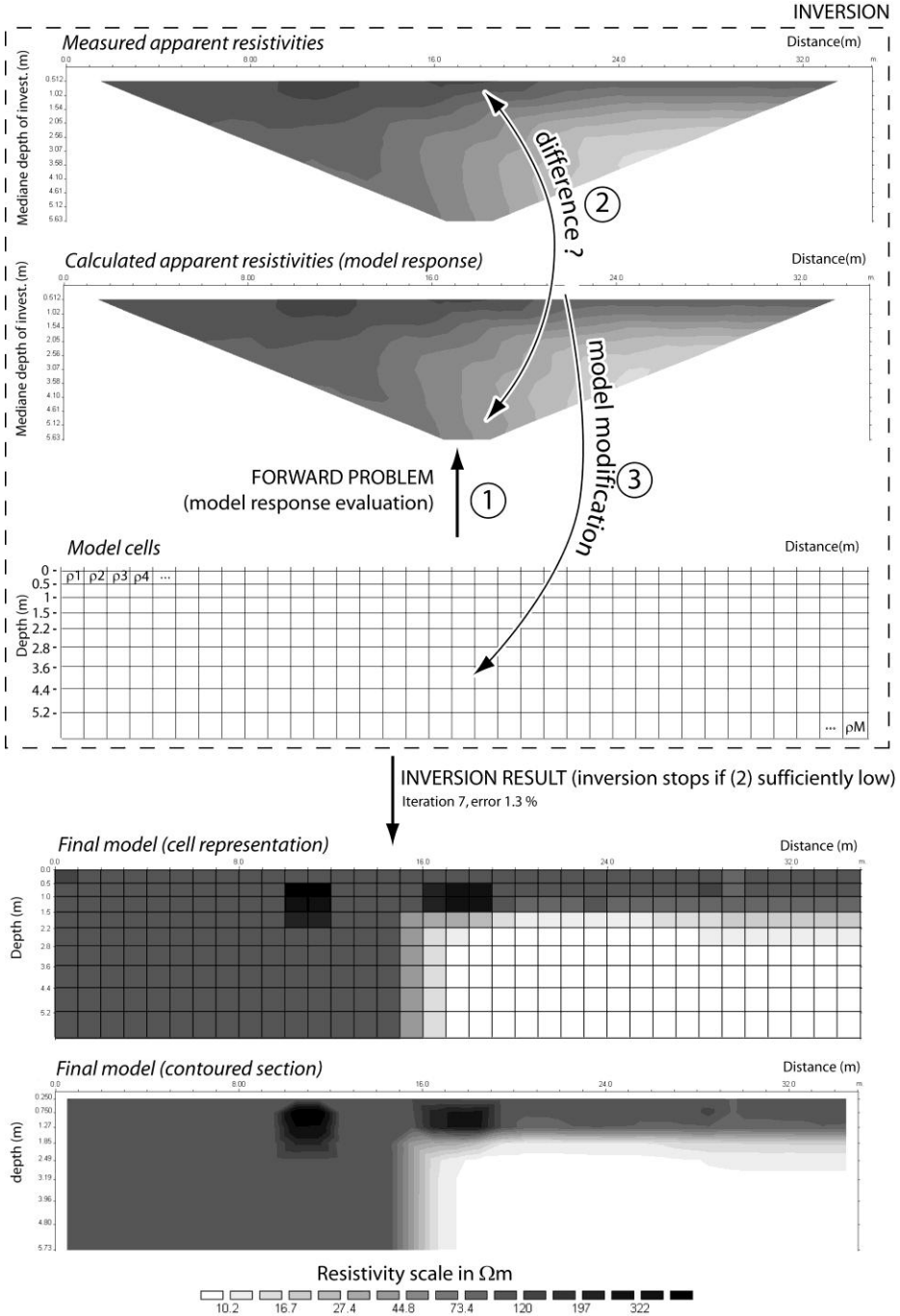


Non-linear Inverse Problem

Current distribution



Electric tomography inversion



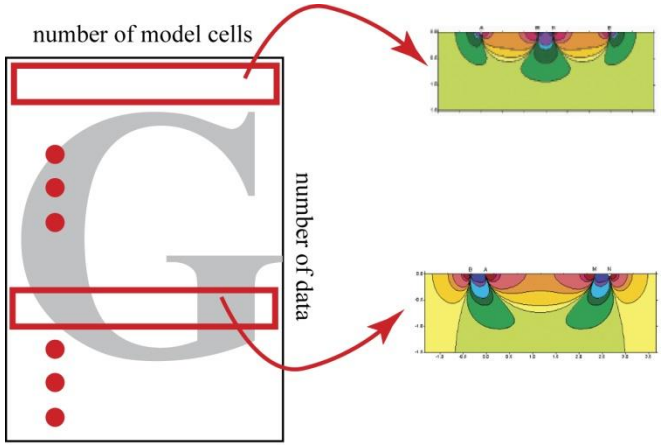
$$\Delta\mathbf{m} = \left[\mathbf{G}^{\mathbf{T}}\mathbf{G} + \mathbf{W}\right]^{-1} \mathbf{G}^{\mathbf{T}}\, \Delta\mathbf{d}$$

with:

$$\Delta\mathbf{m} = \log(\rho)$$

$$\Delta\mathbf{d} = \log(\rho_a^{meas}) - \log(\rho_a^{calc})$$

$$G_{ij} = \frac{\partial g(m)_i}{\partial m_j}$$ is the sensitivity matrix

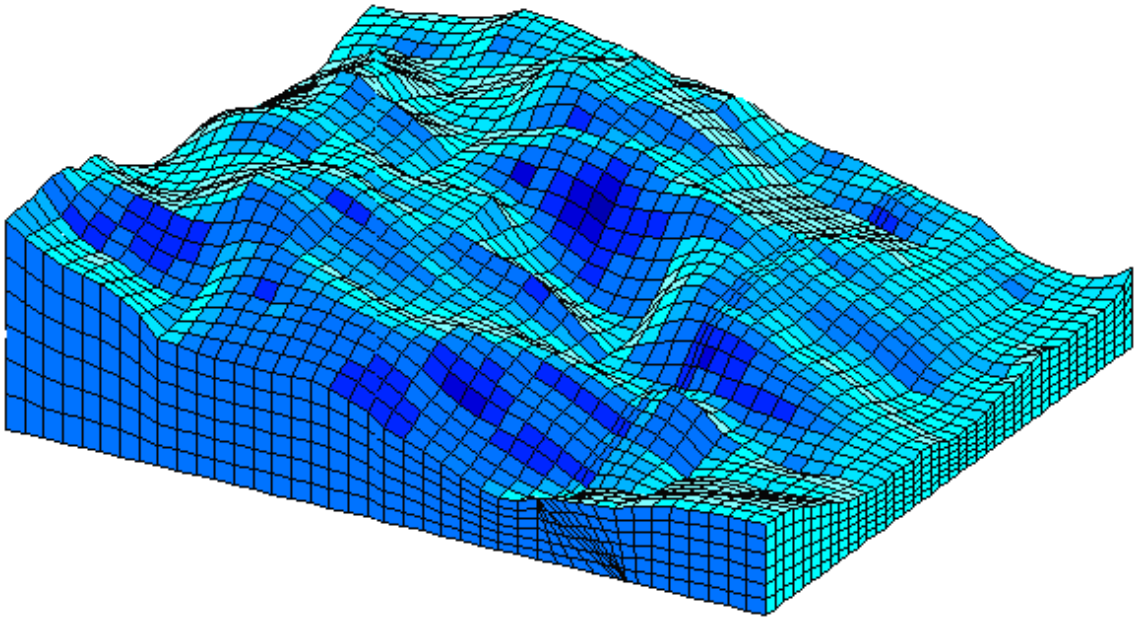


Laurent Marescot, 2010

Modified from Marescot, 2006, Bull. Soc. vaud. Sc. nat. 90.1: 23-40

Complex Model Geometries

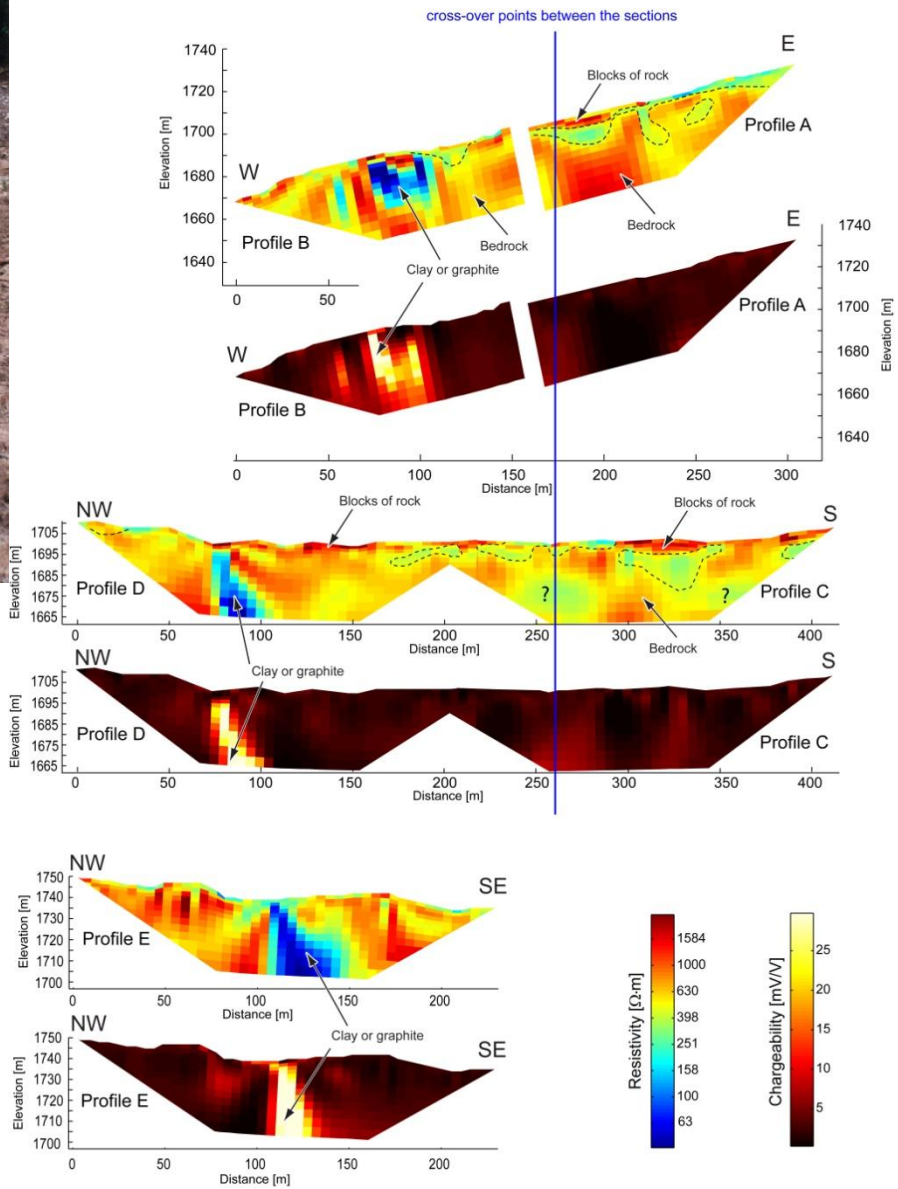
Finite Element Method



Slope Instabilities

Source: Marescot et al., 2008, Engineering Geology 98

Slope Instabilities



Laurent Marescot, 2010

Non-linear Inverse Problem

Key Concepts

- Iterative solution!
- Large Computing may be required
- **G** is a partial derivative matrix (cannot be pre-evaluated!)
- Mixed-determined inverse problem:

$$\Delta\mathbf{m} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon\,\mathbf{I}]^{-1}\,\mathbf{G}^{\mathbf{T}}\,\Delta\mathbf{d}$$

Final notes:

- ε is called the damping factor and is sometimes written λ
- The value of the damping factor is usually decreased at each iteration

Further Reading

The following documents were used to prepare this presentation:

More information on non-linear inversion with examples:

MARESCOT L., *2003. A weighted least-squares inversion algorithm: application to geophysical frequency-domain electromagnetic data. Bull. Soc. vaud. Sc. nat. 88.3: 277-300.*

More information on geoelectrical tomography:

MARESCOT L., 2006. Introduction à l'imagerie électrique du sous-sol. *Bull. Soc. vaud. Sc. nat. 90.1: 23-40.*

These documents can be downloaded at: http://www.tomoquest.com